Electricity

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Electricity

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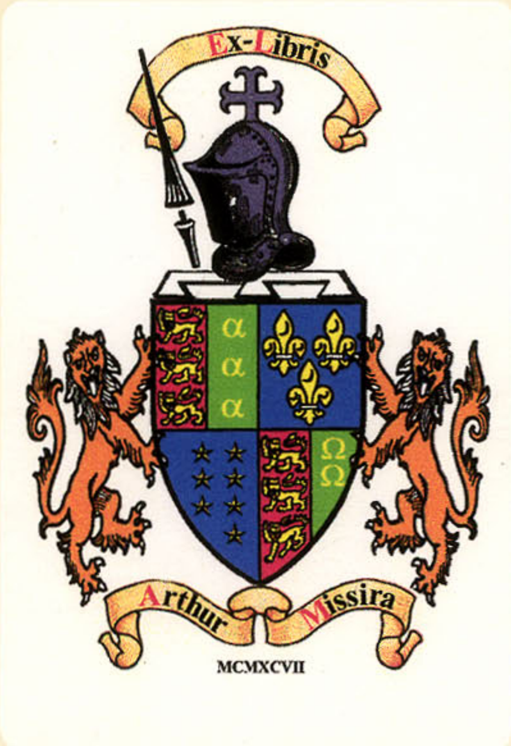

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ELECTRICITY

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PREFACE

THE author of a book on electricity must decide from the first whether his approach is mainly mathematical or physical; efforts at a satisfactory synthesis of the two have not hitherto proved very successful. The present book is intended to outline from the very beginning a consistent mathematical account of the phenomena of electricity and magnetism. In many respects the field covered is similar to that of Maxwell's classical *Treatise on Electricity and Magnetism*. The present book differs from his in being much shorter, in assuming a working knowledge of vector notation, and in making use where necessary of the atomic viewpoint of modern physics. The introduction of an atomic viewpoint is made possible by the fact that most students are now familiar with the main outline of atomic theory: it is also desirable since a habit of thinking in atomic terms and relating atomic behaviour to macroscopic phenomena should be encouraged as soon as possible.

The insistence that electrical phenomena have their origin in atomic properties leads to one important change from the usual: the theory of magnetism is developed entirely independently of the notion of magnetic poles. Since the basis of all magnetism is some form of atomic current, it is wiser to develop the theory of magnetism from the known laws of interaction of currents; there is no need to introduce the postulate of magnetic poles, except for pictorial purposes. Indeed, since an isolated magnetic pole does not exist, the consistency of the argument is improved if no reference is made to any hypothetical law of force between such entities.

On grounds of space it has been necessary to omit certain items: in particular there are very few details given of the experiments which are needed to test the theory at important points; nor is there any account of applied electricity, such as dynamos, motors, influence-machines, post-office boxes, etc. These may all be found in any standard physical text-book. The theory of the earth's magnetism has been

only lightly touched upon, and electrolysis has been completely omitted, for this, like the theory of electrons, belongs more properly to the field of quantum theory, and is outside the range of this book. But with these omissions, an attempt has been made to build a self-consistent mathematical theory, introducing as few postulates as possible, and capable of explaining all the more familiar phenomena of electrostatics, magnetism and electrodynamics.

A word is necessary in the matter of units. These nearly always cause the student a lot of trouble, and experience has shown that the use of practical units from start to finish does not make the fundamental ideas any clearer. For these reasons the discussion of units and dimensions has been deferred to a final chapter, and very little direct reference to them, except in the broad distinction between the electromagnetic and electrostatic systems, is made in the main body of the text.

No-one really understands any branch of mathematics until he has worked a good many examples in it. Accordingly there are examples, many of them embodying important results, at the end of each chapter. An average student should be able to solve at least half of these, though in some of the others he would require help.

It is a pleasure to acknowledge the guidance that I have received from Professors E. T. Copson and G. S. Rushbrooke, who have enabled me to remove some errors and have pointed out several obscurities: to them, and to my wife who has helped me with the preparation of the manuscript, I offer my grateful thanks.

Yet this book would be incomplete without a reference to my former teacher, Mr E. Cunningham, of St John's College, Cambridge, who first showed me how beautifully vector methods fitted the subject of electricity. Much of what is best in this account must be attributed, directly or indirectly, to his influence.

King's College, London
January, 1948

PREFACE TO SECOND AND LATER EDITIONS

The only significant alteration in these editions is the addition of a short note at the end of Chapter XIV concerned with certain similarities between the fundamental electromagnetic vectors.

I am grateful to several friends who have enabled me to remove a certain number of inaccuracies in the first edition.

C. A. C.

Mathematical Institute, Oxford
March, 1961

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CHAPTER I

PRELIMINARY SURVEY

§ 1. Electrostatics

THE fact that a piece of amber, when rubbed, will attract small particles of matter, was known 2500 years ago by Thales of Miletus (640-548 B.C.). From this simple experimental fact has developed the whole science of electrostatics, that is, the properties of electricity at rest. Indeed, the very word electricity is derived from the Greek word for amber. During the years since Thales, and especially in the last 150 years, more experimental knowledge has been accumulated. This knowledge has seldom been obtained in the most systematic order; so our best policy in this book will not be to report the various experiments stage by stage as they were first made, but rather to start with a general survey of our present knowledge. In later chapters we shall from time to time briefly refer to the crucial experiments needed to establish or confirm each particular point of the theory.

We know now that electricity consists of two kinds—positive and negative charges. Like charges repel each other, but opposite charges attract. If it were not for this latter fact our material universe would not hold together at all. The smallest negative charge which it is possible to obtain is that of the ordinary electron, discovered and measured for the first time by J. J. Thomson in 1897. The smallest positive charge has the same numerical value, and is found on the proton; the same charge exists on the positive electron, a much lighter particle than the proton, but this particle is not stable and we shall not need to discuss

it. All charges are integral multiples of these fundamental units; but each unit is so very small that in any common electrical measurement the discreteness of electric charge will not affect us, and we may suppose that a given charge may be allowed any arbitrary numerical value. The smallness of the electronic unit in relation to ordinary measurements may be shown by the fact that in a 60-watt lamp at 200 volts approximately 2×10^{18} electronic units of charge flow along the filament per second. The masses of the electron and proton are extremely small, that of the electron being about 9×10^{-28} grams: the proton is about 1837 times as heavy. Neither the electron nor the proton is a strict mathematical point. One of the most important of the unsolved problems of electricity is the precise nature and size of these elementary particles; it is usual to assign a radius to them of the order of 10^{-13} cms. Since we cannot measure any distances as small as this, it will be quite in order for us to regard our charges as points, and we shall therefore refer, when necessary, to point charges.

Matter, as we understand it, consists of atoms: that is, of positive and negative charges associated together in small structures of about 10^{-8} cms. diameter. The positive charge rests on the heavy part, or nucleus, of each atom, and the negative charge is in the form of electrons that move round the nucleus in much the same way as planets round the sun. The precise manner in which this motion takes place need not concern us, and belongs to atomic theory. All that we need to know is that their motion can be somewhat changed, so that the atom becomes distorted, if the right kind of external force is applied. Normal matter consists of neutral atoms, i.e. atoms in which there is no excess of positive or negative charge. In 1 c.c. of an ordinary solid there are about 10^{23} such atoms. (In diamond, for example, the number is 4×10^{23}.) But some of the atoms may have too few, or too many, electrons to balance the positive nucleus, and then we have net positive or net negative charges present. A study of the forces that these exert on each other, or on any neutral matter that may be present, forms the content of Chapters II-IV; this may be said to represent the science of electrostatics proper.

But this brings us to an important distinction. In the theory of electricity, in contrast with atomic physics, we are not primarily concerned with the forces exerted by one atom, or one electron, on another atom, since the forces and distances involved are far too small for us to measure individually in the laboratory. It is rather with the bulk effect, in which a large number of electrons and atoms are involved, that we shall be concerned. Thus if the smallest mass we can conveniently measure is taken to be $\frac{1}{10}$ milligram, this would represent no less than 10^{23} electrons, or between 10^{18} and 10^{20} atoms, depending on the substance we are using. The distinction that we are making is between the **microscopic** and **macroscopic** points of view. In the microscopic point of view we deal with individual atoms and electrons, which are the field of atomic physics and quantum theory. In the macroscopic point of view we average these forces over the large number of atoms in a tiny measurable volume; for that purpose it is often quite fair to neglect the peculiar individual atomic and inter-atomic effects revealed in a microscopic survey. For example, just as we referred earlier to point-charges, thereby neglecting the internal structure of an electron or proton, so also for many purposes we shall neglect the atomicity, or "graininess," of matter itself.

The question will now be asked: if we are not to take into account this detailed structure of matter, but are to use averages in which we have effectively smoothed it out to become homogeneous and continuous, what is the advantage of introducing the microscopic, or atomic, point of view at all? The answer is twofold. First, the microscopic viewpoint throws light on the fundamental physical processes; this enables us to view our subject as one whole and means that we shall not have to introduce from time to time apparently unrelated physical assumptions, for we shall see how our macroscopic equations arise quite naturally

from simple microscopic properties of the atom and the electron. This is particularly important (see § 3) in discussing the relation between electric currents and magnetism. Secondly, the microscopic point of view prepares us for the more intimate detailed study of these atomic processes which is necessary if we are to understand fully the nature of our physical world.

§ 2. Electric currents

The study of electrostatics, or charges at rest, leads naturally to a study of electric currents, or charges in motion. The current may be caused by movement either of the positive or negative charges, or of both. Thus in a discharge tube positive ions (that is, atoms with an excess of positive charge) move in one direction, and electrons carrying a negative charge move in the opposite direction. It is possible by bending the beams magnetically to separate the two, and obtain a current composed solely of positive or negative charge. On the other hand, in metals, such as a copper wire, the charge is carried entirely by electrons. It makes no difference to our formulation of the laws of current flow, as we develop it in Chapter V, which type of carrier is bearing the charge, for in all cases the current is measured by the rate at which the charge flows, i.e. the net amount crossing unit area in unit time. The direction of the current is the direction in which the mean drift, or flow, of charge is taking place.

The distinction which we made in § 1 between microscopic and macroscopic measurement is important here. For on the microscopic point of view the charges are moving in all directions with all possible speeds ; but on the macroscopic point of view, in which we consider merely the average motion of the charges within a very small volume, we determine a mean drift velocity, the magnitude and direction of which measure the electric current. The situation is rather like the flow of gas down a tube. According to the

Kinetic Theory, the various particles of gas have all possible velocities in all directions ; but the mean velocity lies along the direction of the tube, and for most purposes we may forget the random distribution of velocities and suppose that each particle of gas has, in fact, this mean " drift " velocity of flow down the tube.

For purposes of discussing the flow of current all substances may be placed in one of two categories—**insulators** and **conductors.** An insulator (e.g. amber, glass, shellac) is a substance in which it is practically impossible to cause any current to flow. The explanation is simple, for in these substances all the negative charges (or electrons) are firmly attached to corresponding positive charges. As we cannot easily separate them it follows that no net flow of charge can take place. A conductor, on the other hand, is a substance in which a certain number of electrons (or negative charges) are easily separated from their associated positive charges, and one or both can move under the influence of a force of the right kind. Thus in **metals,** such as copper or tungsten, there are a certain number of electrons, called **metallic,** or **free,** or **conduction electrons,** which are able to flow freely through the material and give rise to the current, while the positive charges remain fixed. But in **electrolytes,** such as the dilute sulphuric acid in an ordinary accumulator, each particle, or molecule, of electrolyte spontaneously separates into positively and negatively charged parts which can move independently of each other. In the case of sulphuric acid, protons move in one direction carrying a positive charge, and sulphate ions move in the opposite direction carrying a negative charge ; the total current is the sum of these two separate currents.

§ 3. Magnetism

We have seen that a current may be measured by the quantity of charge flowing in unit time. This counting of charge, which is made with an electrometer, provides us with

an electrostatic measure of current, and the result would naturally be expressed in electrostatic units, generally abbreviated to e.s.u.; we shall have more to say about these units later. But there is another entirely different way of measuring current; for when charges are flowing we discover a whole series of new phenomena, to which we give the name magnetism. In actual practice it is these magnetic, or more properly electromagnetic, effects that are most commonly used to measure currents, and in such cases our result will be expressed in electromagnetic units and written e.m.u. The same current may be measured in both units, and the relation between them, or, which is the same, the ratio of the corresponding units, is a matter of prime importance. We shall see in Chapter XIII how Maxwell was able to use its known value to show that light waves were essentially an electromagnetic phenomenon.

These magnetic effects, however, had been known in another connection for a very long time. Thus Lucretius mentions that certain mineral ores such as loadstone have the power of attracting small pieces of iron placed near them, and one of the earliest attempts at a perpetual motion machine makes use of this attraction. These forces were called magnetic forces, and the attracting materials were called permanent magnets. From this beginning there developed, quite independently of electrostatics, the science of magnetism, the study of which gained great importance when it was realised that the earth itself behaved like a large permanent magnet. It was the discovery of Oersted (1820) and Faraday (1831), that the same magnetic effects are produced by currents as by permanent magnets, which related the two hitherto distinct sciences. If electric currents are able to produce the same effects as permanent magnets it is natural to enquire whether the so-called permanent magnetism may not be due to currents of some form or other; and indeed it was not long before Ampère proposed the hypothesis that each elementary constituent of matter (as we should say, each atom) was really a minute electric

current. This hypothesis is now accepted. If we go back to our earlier discussion of the atom we can soon see how this comes about, and what is the nature of Ampère's minute currents. We have seen that in an atom electrons move in orbits round the central nucleus; we have also seen that when charges move there is an electric current flowing. Combining these two facts it follows that in each atom there are indeed tiny electric currents. It may happen—and in fact very often does—that in each atom there are pairs of electrons moving in opposite directions, and in such a case the atomic currents cancel so that the atom is not a permanent magnet. It may equally well happen that although each atom is itself magnetic, these magnets are arranged in a given block of material in random directions; then again the substance is not a permanent magnet. But it may be that a majority of these tiny currents are oriented in the same direction, and then we do have a permanent magnet. In the case of substances which are not permanent magnets it is possible, by using the right kind of force, to alter the orbits of the electrons and in this way we may induce a temporary magnetism which vanishes when the disturbing force is removed. This is the phenomenon of **induced magnetism.** Thus the difference between substances which are or are not permanent magnets is not that they are made of essentially different material, but rather that with permanent magnets we have no way (or at any rate no simple way) of destroying the co-operative effect of the separate atoms, whereas with non-permanent magnets this co-operation is solely the result of forces exerted from outside, and automatically disappears when the force is removed.*

Our study of the magnetic effects of currents and the

* We have spoken of an electron in its orbit as being equivalent to a minute current. But there is another type of current, known to atomic physicists as spin, e.g. electron-spin or nuclear-spin, which we have not mentioned. It is good enough for our purposes to regard this as a flow of current inside a single electron or nucleus, giving rise to magnetic effects which may or may not average out to zero, similarly to the electron orbits previously described.

properties of permanent magnets occupies Chapters VI-VIII. In olden days, following the historical sequence that we have already indicated, it was usual to develop the subject of magnetism quite separately from electrostatics, starting from the existence of permanent magnets. The two subjects were finally united through the phenomena of electromagnetism, rather in the way that the lintel of a door rests upon and links together the two door-posts. But we shall find it best to proceed from the electrostatic properties of currents to their electromagnetic properties, and this will then lead us naturally to our discussion of permanent magnetism. The great advantage of such a procedure is that every part of our scheme is immediately related to and follows from our knowledge of the structure of the atom, as we have indicated earlier in this section.

It turns out that there is a close parallelism between the laws of electrostatics and magnetism, so that the same type of mathematical analysis can be employed to solve problems in either field. We therefore break off, in Chapters IX and X, to discuss in detail a selected number of such problems, and to illustrate the technique required in their solution.

§ 4. Electrodynamics

We have so far been mainly concerned with steady currents. When currents do not remain constant we have to distinguish two cases, depending on whether the change is slow (known as quasi-steady currents) or rapid. In the first of these, associated particularly with Lenz and Faraday, we discover that if by any means we try to change the current flowing in any circuit then our apparatus behaves as if it were trying to prevent such change, i.e. it sets up new systems of currents whose effect, by themselves, is to counteract the original change. This is the phenomenon of **electromagnetic induction** which, in its developed form, underlies all the construction of dynamos, transformers and electric motors. Chapter XI is devoted to a study of these quasi-steady currents in general terms. Chapter XII outlines some of the extremely important applications of induction in the alternating current theory of electrical circuits. This is part of the whole field of wireless telegraphy.

We have by this time brought most of the common electric effects within the power of our equations, but not all. It was the peculiar genius of Maxwell that he recognised a previous omission. We have seen that under certain circumstances an atom may be deformed so that the positive and negative charges are slightly separated. While this separation is taking place there is a very small flow of charge, i.e. a small current. If the change is extremely rapid and the negative charge fluctuates from one side to the other of the positive charge then an oscillating current may be said to flow. This is one part of Maxwell's Displacement Current, and we shall see in Chapter XIII that although for quasi-steady or steady systems this new term is not effective, for sufficiently rapidly varying currents it often becomes dominant. When it is included we are led to formulate the general equations of the electromagnetic field, usually known as Maxwell's Equations. It is upon these equations that all later development rests. These equations show for the first time the possibility of electromagnetic waves, and by comparison with the velocity of light we infer that light itself is an electromagnetic phenomenon. A whole new field of experience, including such diverse phenomena as the laws of reflection and refraction and the energy radiated from aerials is brought within our scope. The equations themselves reveal a surprising symmetry with respect to electrostatic and magnetic effects. We have seen that electric charges in motion produce the same effect as magnets at rest; it now appears that magnets in motion produce the same result as charges at rest. The link between electrostatics and magnetism is not only that the mathematical technique is the same for both; rather this latter fact is itself the expression of an organic unity—they are both parts of the complete study of electrodynamics.

This is about as far as our macroscopic theories can profitably carry us. Further progress depends upon a more detailed study of microscopic, or atomic, properties. Such studies have been made, but they introduce us to two essentially new ideas—the quantum theory and the wave nature of matter. As such they are outside the scope of this book.

CHAPTER II

ELECTROSTATICS

§ 5. Law of Force

THE study of electrostatics is based primarily upon the experiments of Cavendish using a charged sphere, and of Coulomb using a torsion balance. These show that if two charges e_1 and e_2 are a distance r apart *in vacuo* the force between them is proportional to e_1e_2/r^2. If the charges are of like sign the force is repulsive; if different, it is attractive. This is known as the **Electrostatic Inverse Square Law.**

It is impossible to prove that the law of force is exactly that of the inverse square. The original experiments of Cavendish showed that if the force was proportional to $1/r^n$, then n lay between $2 \pm 0{\cdot}02$. Recent researches show us that n lies between 2 ± 10^{-9}. We may therefore use the inverse square law with complete confidence, provided that we do not apply it in conditions where the dimensions differ essentially from those under which it has been determined. At distances comparable with the radius of the electron (10^{-13} cms.) we cannot be sure that the law will still hold. Since our interest lies in macroscopic rather than microscopic electrical properties, we shall be safe in accepting the validity of the law throughout this book.

The c.g.s. **electrostatic unit** of charge (e.s.u.) is defined to be such that the force is exactly equal to e_1e_2/r^2 dynes when r is measured in cms. Two unit charges repel each other with a force of 1 dyne at a distance of 1 cm. *in vacuo*. The question of the force in other media is left till Chapter IV.

§ 6. Potential and field for a single charge

If a unit charge is placed at some point P in the presence of a series of given fixed charges, it will in general experience a force. This force is defined to be the **electrostatic field,** or simply the **electric field,** at P. Since it has magnitude as well as direction the field is a vector quantity, denoted by $\mathbf{E}$,* with components E_x, E_y, E_z. The field $\mathbf{E}$ is of course quite independent of the unit charge at P, which is simply used to test and measure it.

Coulomb's law enables us to calculate the electric field due to a single point charge e at O (Fig. 1). For if $OP = r$, a unit charge at P will experience a force e/r^2 along the line OP. The field at P is therefore of magnitude e/r^2 and in the direction OP. If $\mathbf{r}$ denotes the vector OP, we can write

$$\mathbf{E} = \frac{e\mathbf{r}}{r^3} \quad . \quad . \quad . \quad . \quad (1)$$

Now let us suppose that the unit test charge is moved from A to B (Fig. 2) still under the influence of the fixed charge

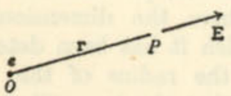


Fig. 1

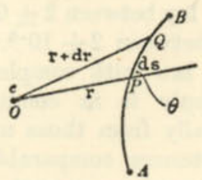


Fig. 2

e at O. Let us calculate the work that is done in this process. In going a distance $\mathbf{ds}$ from P to Q the field exerts a force $\mathbf{E}$, and therefore does work

$$\mathbf{E}.\mathbf{ds} = \frac{e}{r^2}\,ds\cos\theta = \frac{e\,dr}{r^2},$$

* Clarendon type will always be used to denote vector quantities.

where θ is the angle between OP and PQ. The total work done by the field in moving the charge from A to B is therefore

$$\int_A^B \frac{e\,dr}{r^2} = \frac{e}{r_A} - \frac{e}{r_B}.$$

It is often more convenient to think of the work we do against the field rather than of the work done by the field. Thus

$$\text{work we do} = -\text{work done by field} = \frac{e}{r_B} - \frac{e}{r_A} \quad . \quad (2)$$

Notice that this depends only on the positions of A and B and not on the path connecting them. All paths from A to B involve us in the same expenditure of work. If this were not so, and there was a certain path 1 which involved us in doing less work than another path 2, then by going out along 1 and back along 2 we should recover more work along 2 than we had expended along 1. By continuing this process we could obtain an indefinite amount of work or energy. We know that this is impossible in all natural phenomena, in which the fields of force are said to be **conservative.**

The quantity occurring on the right of equation (2) is called the **difference of potential** between A and B, and is written $V_B - V_A$. Thus the difference of potential between two points represents the work we have to do to move a unit test charge from the one to the other. We generally define the absolute value of the **potential at a point** by the condition that if A is at infinity $V_A = 0$. V_B then represents the work we do in bringing unit charge from infinity to B. From (2) we see that $V_B = e/r_B$. Dropping the suffixes we may write for the potential due to a charge e:

$$V = e/r \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If our test charge has magnitude e' instead of unity, Coulomb's law shows that the force on it is merely multiplied

by e'; therefore the work we do from A to B is also multiplied by e', and may be written:

$$\text{Work we do from } A \text{ to } B = e'(V_B - V_A). \qquad (4)$$

We can obtain a very important formula by considering the difference of potential dV between P and Q in Fig. 2. Thus

$$\begin{aligned} dV &= V_Q - V_P \\ &= \text{work we do on unit charge from } P \text{ to } Q \\ &= -\text{work done by field} \\ &= -\mathbf{E}.\mathbf{ds} \end{aligned} \qquad (5)$$

In Cartesian co-ordinates this becomes

$$dV = -(E_x dx + E_y dy + E_z dz), \qquad (6)$$

leading to the important relations

$$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y},\ E_z = -\frac{\partial V}{\partial z}. \qquad (7)$$

These are conveniently summarised in the vector equation

$$\mathbf{E} = -\text{grad } V = -\nabla V. \qquad (8)$$

It is sometimes convenient to write (5) in the integrated form

$$V_B - V_A = -\int_A^B \mathbf{E}.\mathbf{ds}. \qquad (9)$$

§ 7. Field due to several charges

The torsion balance experiments of Coulomb show two other extremely important facts. The first is that both the field and the potential are proportional to the charge e which causes them; thus if we double e we also double V and $\mathbf{E}$. This is indeed implicit in our formulation of the law of force. The second fact is that if instead of one charge e we have two or more distinct charges, then their effects are simply additive. Thus two fixed charges e_1 and e_2 exert on a third charge e whose distances

from them are r_1 and r_2 a force equal to the sum of the two forces ee_1/r_1^2 and ee_2/r_2^2 added vectorially in the usual manner for forces. These two facts constitute the **principle of superposition**, which states that for any given system of charges, the final force and potential are the same as if we simply superposed the separate forces and potentials. It follows that our discussion, at the end of the last section, of the potential V and field $\mathbf{E}$ due to a single charge can be applied to a system of several charges $e_1, e_2, \ldots$ In fact, V still represents the work we do in bringing unit charge from infinity to any point, and $\mathbf{E}$ is the force on the unit charge. Equations (5) to (9) are valid just as they stand, and if $V_1, V_2, \ldots$ are the potentials due to each separate charge,

$$V = V_1 + V_2 + \ldots = \Sigma \frac{e}{r} \qquad (10)$$

Similarly

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots = \Sigma \frac{e\mathbf{r}}{r^3} \qquad (11)$$

$\mathbf{r}$ is, of course, the vector from one of the given charges to the point at which $\mathbf{E}$ is required.

We can always investigate the field due to a set of charges by putting a small charge e' at various points. The force on it is $e'\mathbf{E}$, from which we determine the magnitude and direction of $\mathbf{E}$. We sometimes refer to this operation as **mapping**, or **exploring**, the field; the charge e' is called the **test charge.** Having found $\mathbf{E}$ we can get V from the equation $\mathbf{E} = -\text{grad } V$, or we can measure V directly by the work $e'V$ required to bring the test charge from a long distance away to the chosen point.

In mathematical problems it is nearly always preferable to use V rather than $\mathbf{E}$. This is because V is a scalar quantity whereas $\mathbf{E}$ is a vector quantity, and calculations with vectors are usually more involved than with scalars.

It sometimes happens that our fixed charges are not all separate and distinct, but that they are distributed through

a given volume or on given surfaces. We speak of a volume density ρ if the charge in volume dv is $\rho\, dv$; and of surface density σ if the charge on an element of surface dS is $\sigma\, dS$. It is quite obvious that the appropriate modification of (10) that deals with volume and surface distributions is

$$V = \int \frac{\rho\, dv}{r} + \int \frac{\sigma\, dS}{r} \quad . \quad . \quad . \quad . \quad (12)$$

If ρ and σ are known this gives us a way of calculating V. Since $\mathbf{E} = -\text{grad}\, V$, we soon get $\mathbf{E}$ also. Unfortunately, apart from a few special cases, the integration is usually very difficult.

§ 8. Equipotentials and Lines of Force

According to (10) and (12) at any point P there is a definite value for the potential. Consider all points for which the potential has the same value V. As a rule these will lie on a surface. We call this an **equipotential surface.** By choosing different values of V we obtain a whole series of equipotential surfaces, or simply, equipotentials. Since each point can have only one potential, there is one and only one equipotential through any point P. If the potential at any point (x, y, z) is given by the function $V(x, y, z)$, then the family, or system, of equipotentials has the equation

$$V(x, y, z) = \text{constant} \quad . \quad . \quad . \quad (13)$$

By taking different numerical values for the constant we obtain all the various members of the system of surfaces.

Let us take the special case of a solitary charge e at the origin. Then according to (3) the potential is e/r, so that the equipotentials are the surfaces $e/r = \text{constant}$, i.e. concentric spheres around the origin.

Similarly if we take the case of two separate charges e_1 and e_2, the potential at a point distant r_1 and r_2 from the

charges is $e_1/r_1 + e_2/r_2$. The equipotential surfaces are therefore obtained by rotating the set of Cassini ovals

$$\frac{e_1}{r_1} + \frac{e_2}{r_2} = \text{constant}$$

about the line joining the two charges.

The normal at any point on the equipotential surface $V(x, y, z) = \text{constant}$, will have direction cosines proportional to $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}$. But from (7) $E_x = -\frac{\partial V}{\partial x}$, etc., so the direction cosines of $\mathbf{E}$ are also proportional to $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}$. We deduce the important theorem that the direction of $\mathbf{E}$ at any point is normal to the equipotential surface through that point.

This last theorem suggests a new way of mapping the field $\mathbf{E}$. Suppose we draw a set of lines, which will usually be curved, such that at any point on one of these lines the direction of $\mathbf{E}$ coincides with the tangent to the line at this point. These lines are known as **field lines, or lines of force.** Through any given point there is usually just one line of force, and the set of lines so obtained are the orthogonal trajectories of the equipotential surfaces; that is, every field line cuts every equipotential surface at right angles.

We can soon write down the differential equations of the lines of force. Since the direction of the line is the same as the direction of $\mathbf{E}$, the lines must be defined by $dx : dy : dz = E_x : E_y : E_z$. I.e.,

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} \quad . \quad . \quad . \quad . \quad (14)$$

In the particular case of a solitary charge at a point O, the lines of force are evidently radii drawn from O in all directions. There is one and only one radius through any point, and these lines all cut the equipotentials, which are concentric spheres around O, at right angles.

We can give another interpretation of lines of force. For if a charge e is placed at a point where the field is $\mathbf{E}$, it will experience a force $e\mathbf{E}$. If we release it from rest it will start to move in the direction of the force, that is, of $\mathbf{E}$, which is along the line of force. On account of its mass and momentum, however, it does not follow that the charge will continue to move along the line of force as it gathers speed.

Further, since the direction of $\mathbf{E}$ is along the line of force, the fundamental equation $\mathbf{E} = -\text{grad } V$ may be written

$$E = -\frac{\partial V}{\partial s}, \quad . \quad . \quad . \quad . \ (15)$$

where $\frac{\partial}{\partial s}$ denotes differentiation along the line of force. This equation also shows us that the positive direction of a line of force is the one in which V decreases. As we proceed along any particular line of force the potential continually falls.

Next let us take any small closed contour C (Fig. 3), and draw the lines of force that pass through every point of C. These lines form a thin tube whose cross-section may vary along its length, but whose generators at any point are all effectively parallel. Following Faraday we call this a **tube of force**, and define the **strength of a tube** as the product of the magnitude of $\mathbf{E}$ and the normal sectional area. As we shall see shortly, the strength of a tube remains constant all along its length. A **unit tube** is one whose strength is unity. Tubes of force, like lines of force, are everywhere parallel to the direction of $\mathbf{E}$, and the potential falls continuously as we proceed in the positive direction along them. Two lines of force, or two tubes of force, can never meet, for that would involve two distinct directions of $\mathbf{E}$ at the common point. An apparent exception to this rule is where $\mathbf{E} = 0$, since at such points (14) shows that $dx : dy : dz$ is in-

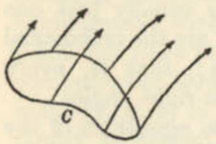


FIG. 3

determinate, so that the field lines themselves are indeterminate. If $\mathbf{E} = 0$ at a point P, it means that a charge placed at P would experience no resultant force, and would therefore be in equilibrium. For this reason we refer to such points as **neutral points**, or **points of equilibrium** of the field. (See § 12, Ques. 5.)

§ 9. Flux

It follows from our definition of tube strength that the number of unit tubes crossing any small area dS drawn perpendicular to the field is $E\,dS$. If dS is represented by the vector $\mathbf{dS}$, this can be written $\mathbf{E}.\mathbf{dS}$, a formula which holds equally well even if dS is not normal to the field. The expression $\mathbf{E}.\mathbf{dS}$ occurs quite frequently, and is called the **flux of E** across **dS**. If we have a larger surface S, bounded by a closed curve C, we may subdivide S into small elements dS; then, by summing the contributions from each element we calculate the total flux across S. Clearly

$$\text{flux of } \mathbf{E} \text{ across } S = \int \mathbf{E}.\mathbf{dS} \quad . \quad . \quad . \ (16)$$

If the surface S is a closed surface, we speak of the flux of $\mathbf{E}$ out of, or into, S. If the direction of $\mathbf{dS}$ is that of the outward normal, then

$$\text{flux of } \mathbf{E} \text{ out of } S = \int \mathbf{E}.\mathbf{dS} \quad . \quad . \quad . \ (17)$$

In (16) the integration is over an open surface, in (17) it is over a closed surface. These two formulæ represent the total number of unit tubes of $\mathbf{E}$ crossing the open surface bounded by C, and emerging from the closed surface S, respectively.

Let us consider more closely the flux of $\mathbf{E}$ across a given surface S (Fig. 4) when the field arises from a single charge e at O.

Then if θ is the angle between an element $\mathbf{dS}$ and the field $\mathbf{E}$,

$$\mathbf{E}.\mathbf{dS} = E\,dS\cos\theta = \frac{e}{r^2}dS\cos\theta = e\,d\omega,$$

where $d\omega$ is the element of solid angle subtended at O by the surface element dS. It follows from (16) that the flux of $\mathbf{E}$ across the whole surface S is $\int e\,d\omega$, or simply $e\omega$, where ω is the solid angle subtended at O by the boundary curve C.

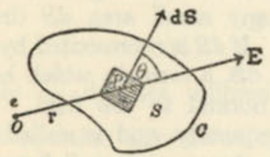


FIG. 4

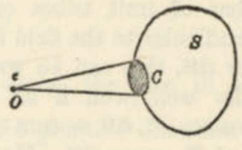


FIG. 5

If the surface S is a closed surface surrounding O, then $\omega = 4\pi$, so that the flux of $\mathbf{E}$ out of any closed surface completely surrounding a charge e is $4\pi e$. This can be expressed by saying that $4\pi e$ unit tubes of force leave any charge e; or alternatively that 4π unit tubes of force leave unit charge.

If the surface S is a closed surface not surrounding O, we may regard it (Fig. 5) as the limit of an open surface when the boundary curve C shrinks to zero. This shows that when the surface is closed, $\omega = 0$, and hence that the flux out of any closed surface not containing the charge e is zero.

It makes no difference to these conclusions if the surface S is re-entrant.

§ 10. Gauss' Law

This brings us to an extremely important result known as Gauss' law. Suppose we have a series of charges e_1, e_2 . . . and we draw a closed surface S which may

surround some or all of them. Then, since the principle of superposition shows that the total field $\mathbf{E}$ is the sum of the contributions from each separate charge, the same is true for the flux of $\mathbf{E}$ out of S. We have just shown that charges outside S give no flux out of S, and each charge inside gives a contribution $4\pi e$. This proves Gauss' law, that the total flux out of any closed surface S is equal to 4π times the total included charge.

We have proved the law on the hypothesis that the charges are each distinct. But we may regard a volume distribution ρ as a series of discrete charges ρdv, and hence the theorem is equally applicable in this case. Gauss' law may be written:

$$\text{flux of } \mathbf{E} \text{ out of } S = \int \mathbf{E}.\mathbf{dS} = 4\pi \times \text{included charge} \quad . \quad (18)$$

Equation (18) takes a particularly simple form in any region where there are no concentrated point charges, but there is a volume distribution ρ. The included charge is $\int \rho dv$, so that $\int \mathbf{E}.\mathbf{dS} = 4\pi \int \rho dv$. But by Green's theorem *

$$\int \mathbf{E}.\mathbf{dS} = \int \operatorname{div} \mathbf{E}\, dv.$$

So
$$\int (\operatorname{div} \mathbf{E} - 4\pi\rho)dv = 0.$$

This equation holds for any volume V, large or small, which implies that $\operatorname{div} \mathbf{E} - 4\pi\rho$ itself must be zero. Gauss' law may therefore be written in the form

$$\operatorname{div} \mathbf{E} \equiv \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi\rho \quad . \quad . \quad (19)$$

* E.g. Rutherford, *Vector Methods*, 1946, p. 74. This important theorem is often called Gauss' theorem; Gauss has historical precedence in its development.

If we combine (19) with the earlier equation (8) for the potential, and remember that

$$\text{div grad } V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \nabla^2 V,$$

we obtain

$$\nabla^2 V = \text{div grad } V = -\text{div } \mathbf{E} = -4\pi\rho \quad . \quad . \quad (20)$$

This equation is known as **Poisson's Equation**; in the particular case of zero charge density ($\rho = 0$) it becomes **Laplace's Equation:**

$$\nabla^2 V = 0 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Equations (20) and (21) are alternative expressions, in the form of a differential equation, of Gauss' law; our method of deriving this equation in p. 21 showed that it depended only on the validity of the inverse square law and the principle of superposition. We may therefore say that (20) and (21) are equivalent to a statement that the law of force between charges is the inverse square, and that electric fields and potentials are additive.

The appropriate solution of (20) has already been obtained; it is, of course, equation (12).

§ 11. Deductions from Gauss' Law

One immediate deduction from (18) is that the potential cannot have a maximum or minimum value except at points where there is a positive or negative charge respectively. For let the potential have a maximum at some point P. Then V decreases in all directions away from P, so that if we draw a small sphere with centre at P, the lines of $\mathbf{E}$ cross this surface everywhere from inside to outside. Thus the flux of $\mathbf{E}$ out of the surface is necessarily positive, showing from (18) that there must be an included positive charge. However small the sphere, there is still charge inside it; this charge must therefore be at P, proving our theorem. A similar argument applies to minimum values of the potential. It does not follow from the above argument that the charge is necessarily a concentrated point charge at P. The conditions

are equally well satisfied by a charge cloud in which the density at P is not zero. For at a point where V is a maximum (but not infinite) $\partial^2 V/\partial x^2$, $\partial^2 V/\partial y^2$, $\partial^2 V/\partial z^2$ must all be negative. So $\nabla^2 V$ is negative, and hence, from (20) ρ is positive.

It also follows that a freely movable charge cannot be in stable equilibrium at a point unoccupied by charge. For with a given system of fixed charges, the potential energy of a movable charge e is eV. Stable equilibrium of e at some point P implies that eV is a minimum at P, and we have just seen that this is impossible unless there is charge at P. This is **Earnshaw's Theorem**: its importance lies in the fact that it shows that no purely stationary system of charges can be in stable equilibrium under their own influence. Hence if our electrical forces hold unchanged in the region of atomic dimensions, it follows that the electrons in an atom or molecule can never be at rest; in fact atoms must be in dynamical rather than statical equilibrium.

Another deduction from Gauss' law is that tubes of force can only begin or end on charges. For suppose a tube of force begins at a point P. Then if we surround P by a small closed surface, there will be more tubes leaving this surface outwards than there are entering it inwards. This means that there is a net flux of $\mathbf{E}$ out of the surface, and hence, from (18), that there is a positive charge at P. In a similar way tubes of force can only end on negative charge.

It is now possible to provide the proof of our statement in § 8, that the strength of a tube of force remains constant along its length. To prove this, consider (Fig. 6) a tube of force with two normal sections dS_1 and dS_2 represented vectorially by $\mathbf{dS}_1$ and $\mathbf{dS}_2$, across which the fields are $\mathbf{E}_1$ and $\mathbf{E}_2$. As the tube does not either begin or end between dS_1 and dS_2 there is no charge included in the closed surface formed by dS_1, dS_2 and the lines of force which form the generators of the tube. Hence there is no net flux of $\mathbf{E}$ out of this surface. No tubes can cross the curved boundary since this surface is itself the boundary of a tube, and hence the flux across dS_1 equals the flux across dS_2. This means that

$\mathbf{E}_1.\mathbf{dS}_1 = \mathbf{E}_2.\mathbf{dS}_2$. But by definition $\mathbf{E}.\mathbf{dS}$ is the strength of a tube, so that the strength does remain constant along its length.

As another example of the use of Gauss' law, let us consider the field due to an infinite layer of charge, with constant density σ, lying on the plane $ABCD$ (Fig. 7A). In the first place, by symmetry, the lines of force are everywhere directed normally away from the layer of charge. Consider,

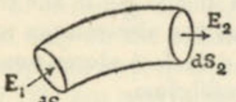


FIG. 6

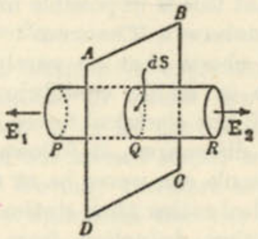


FIG. 7A

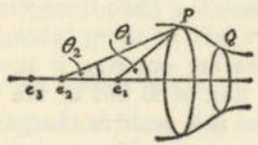


FIG. 7B

therefore, the flux of $\mathbf{E}$ out of the right cylinder PQR, whose generators are all perpendicular to $ABCD$, and whose section is dS. Then, in the notation of the figure, the flux of $\mathbf{E}$ out of the end R is E_2dS, and out of the end P is E_1dS. There is no flux out across the curved surface, and the total included charge is σdS. So Gauss' law (18) tells us that

$$(E_1+E_2)dS = 4\pi\sigma dS,$$

i.e.

$$E_1+E_2 = 4\pi\sigma.$$

This equation is valid whatever the distances PQ, QR. So we have proved that the field is everywhere constant in

magnitude and directed away from the layer of charge. If there are no other charges present, then by symmetry $E_1 = E_2$, so that

$$E_1 = E_2 = 2\pi\sigma \quad . \quad . \quad . \quad . \quad (22)$$

Surfaces such as the above cylinder PQR, which we introduce and to which we apply Gauss' law are often called **Gaussian surfaces.** They are, of course, surfaces in a geometrical, and not a material, sense.

Lastly, let us consider the equations of the lines of force when there is a series of collinear charges $e_1, e_2, \ldots e_n$. Let PQ (Fig. 7B) be an element of a line of force. Then since the field must have cylindrical symmetry about the line of charges, we can rotate PQ about this line and form a closed surface such that no lines of force cross its curved part. Consequently the flux of $\mathbf{E}$ across the plane circle at P is the same as that across the plane circle at Q. According to § 9 the total flux from left to right across P is $e_1\omega_1+e_2\omega_2+\ldots$, where $\omega_1, \ldots$ are the solid angles subtended at $e_1, \ldots$ by the circle. But $\omega_1 = 2\pi(1-\cos\theta_1)$, and so the function $\Sigma 2\pi e(1-\cos\theta)$ has the same value at P and at Q. Now $\Sigma 2\pi e$ is constant, and so the lines of force are defined by the equation

$$\Sigma e \cos\theta = \text{constant} \quad . \quad . \quad . \quad . \quad (23)$$

We have assumed in the above argument that none of the charges $e_1, \ldots$ lie between the sections P and Q. We leave it as an exercise to the reader to show that the equation is unaffected if this restriction is removed, so that (23) defines the lines of force along their whole length.

§ 12. Examples

1. Draw a rough diagram of the lines of force for systems consisting of a charge $2e$ at the origin and either $\pm e$ at some other point. Show that in either case there is just one neutral point, and that with $2e$ and $-e$ the lines of force which separate those lines that go from $2e$ to infinity from those that go to $-e$ leave

the positive charge at right angles to the line joining the charges. Sketch the equipotential surfaces.

2. Obtain an alternative proof of equation (23) which gives the lines of force for a system of collinear charges, as follows. Let P and Q be two adjacent points along a certain line of force; then the direction of the field at P is PQ. Write down the condition that there is no resultant field perpendicular to PQ and show that this is equivalent to the relation $\delta[\Sigma e \cos\theta] = 0$. Deduce (23).

3. Two charges of opposite sign are placed in given positions. Show that the equipotential surface for which $V = 0$ is spherical, whatever numerical values the charges may have. Discuss the apparently exceptional case when the two charges are equal in magnitude.

4. A particle whose charge is $-e$ and whose mass is m is free to move in the plane of the paper. There is also a fixed charge Ze at the origin. Show that if the only force acting on the moving charge is the electrostatic attraction of the fixed charge, then it may describe a circle of any chosen radius r about the origin, provided that its velocity v satisfies the equation $mv^2 = Ze^2/r$. (This result is important in Bohr's theory of the atom.)

5. The point (a, b, c) is a point of equilibrium in a given fixed system of charges, and there is no charge at this point. Expand V in powers of $x-a$, $y-b$, $z-c$ and deduce that in the neighbourhood of (a, b, c) the equipotential surface through this point is in general a cone of the second degree on which it is possible to find three mutually perpendicular generators.

6. A and B are two equal charges $+e$ fixed at the points $(\pm a, 0, 0)$. A third charge $-e$ of mass m revolves around the x axis under the influence of its attraction to A and B. Show that if it describes a circle of radius r, its velocity v is given by

$$mv^2 = 2e^2r^2/(r^2+a^2)^{\frac{3}{2}}.$$

(This was an early model for the motion of an electron in some simple molecules.)

7. Equal charges $+e$ are fixed at the four corners of a square of side $\sqrt{2}a$. A fifth charge $+e$, whose mass is m, is now placed at the centre of the square and is free to move. Show that it is in equilibrium at this point, and that the equilibrium is stable for all small displacements in the plane of the charges, the period of small oscillations in any direction in this plane being $\pi\sqrt{(2ma^3/e^2)}$, but that it is unstable with respect to motion perpendicular to the plane.

8. Electric charge is distributed on an infinite plane surface so that the surface density is σ. P is a point distant a from the plane, and dS is an element of surface whose distance from P is r. Prove that the electric field at P has a component away from the plane equal to $\int a\sigma dS/r^3$. If σ is constant, show that this gives a value $2\pi\sigma$, and deduce that in such a case one half of the field arises from those points of the plane that are less than $2a$ from P.

9. A certain distribution of electric charge is spherically symmetrical about the origin, and the total charge inside a sphere of radius r is $Q(r)$. Prove that the potential $V(r)$ is given by

$$V(r) = \int_r^\infty \frac{Q(r)}{r^2}\,dr.$$

Show that this may be written in the alternative form

$$V(r) = \frac{Q(r)}{r} + \int_r^\infty 4\pi r\rho(r)dr,$$

where $\rho(r)$ is the density of charge at distance r from the origin.

10. A fixed circle is drawn of radius a and a charge e is placed at a distance $3a/4$ from the centre of the circle on a line through the centre perpendicular to the plane of the circle. Show that the flux of $\mathbf{E}$ through the circle is $4\pi e/5$. If a second charge e' is similarly placed at a distance $5a/12$ on the opposite side of the circle, and there is no net flux through the circle, prove that $e' = 13e/20$.

11. Show that at all finite distances from an infinite plane charged on each side with uniform surface density σ, the electric field is of constant magnitude $4\pi\sigma$.

12. Use Gauss' flux theorem to show that there is a change $4\pi\sigma$ in the normal component of $\mathbf{E}$ on crossing a layer of charge of density σ. Deduce that if a line of force crosses a positive layer of charge it is refracted towards the normal.

13. Obtain the result (22) for the field due to an infinite plane layer of charge σ by direct summation of the contribution from each element, as in (11).

14. Electric charge is distributed with constant density σ on the surface of a disc of radius a. Show that the potential at distance x away from the disc along the axis of symmetry is $2\pi\sigma\{\sqrt{(a^2+x^2)}-x\}$. Deduce the value of the electric field, and then, by making a tend to infinity, reproduce the result (22) for the field due to an infinite layer of charge.

15. Electric charge is distributed at constant density e per unit length of an infinite straight line. (This is known as a **line charge** of strength e.) Apply Gauss' law to a cylinder of unit length and radius r coaxial with the straight line and deduce that the intensity E at distance r from the line is $2e/r$. Show that the potential is $-2e\log r +$ constant. Why is it not possible, without more information, to determine the constant?

16. A system of discrete charges is such that the total charge is q and the position of the centroid of charge is G. P is a point at large distance r away from G. By expanding the potential in powers of $1/r$, show that

$$V_P = \frac{q}{r} + \frac{A+B+C-3I}{2r^3} + \dots$$

A, B, C are the second moments (moments of inertia) of the charges about any three perpendicular axes at G, and I is their second moment about the line GP. (This formula is sometimes known as MacCullagh's formula, and is useful in astronomical problems.)

Deduce that at large distances from the charges the equipotential surfaces are effectively concentric spheres with centre G.

17. Two given distributions of charge are separated by a large distance. Their respective centroids are at G and G', and their total charges are q and q'. A, B, C, I are the second moments of the first system of charges about three perpendicular axes through G and about GG' respectively. A similar interpretation holds for A', B', C', I'. By using the result of the previous question, or otherwise, show that the mutual potential energy existing between the two systems is

$$\frac{qq'}{R} + \frac{q'\{A+B+C-3I\}+q\{A'+B'+C'-3I'\}}{2R^3} + \dots$$

where R is the length GG'.

CHAPTER III

CONDUCTORS, DIPOLES AND CONDENSERS

§13. Conductors

Gauss' law enables us very easily to obtain certain properties of the charge distribution in conductors. We remember that a conductor is a substance in which there is no resistance to the movement of charge. Hence if there is any charge at a point inside a conductor there cannot at the same time be any field at that point; for if there was a field there would be a corresponding force on the charge and it would move in the direction of the field. Actually we can soon show that there cannot be any charge at rest inside the conductor. For div $\mathbf{E} = 4\pi\rho$ (§ 10 eq. 19); therefore if $\rho \neq 0$, we conclude that div $\mathbf{E} \neq 0$. Thus $\mathbf{E}$ is not zero and the hypothetical distribution of charge would move in the direction of $\mathbf{E}$. We conclude that if a conductor is charged and no current is flowing, the charge is all on the surface.

Furthermore, the surface of the conductor must be at the same potential everywhere; if it were not so, charges would flow from the places of high potential to those of low potential: this would continue till the whole of the surface was at the same potential.

In this condition there can be no tubes of force inside the conductor; if there were, since there is no charge inside, the tubes would have to start and end on the surface. This is impossible because the potential continually falls along a tube of force and we have just shown that the potential is the same at all points of the surface. Since there are no tubes of force inside, there is no field inside. So $\mathbf{E} = 0$

inside a conductor, and the potential has the same constant value at all points of the conductor.

Since the surface of the conductor is an equipotential, it follows that the direction of the field just outside the surface is perpendicular to the surface, i.e. the lines of force leave the surface normally. Thus if E_s is the component of $\mathbf{E}$ in any direction tangential to the surface

$$E_s = 0 \quad . \quad . \quad . \quad . \quad (1)$$

The normal component E_n may be calculated in terms of the surface charge density σ by applying Gauss' law to a small cylinder (Fig. 8) with cross-section dS, whose generators are perpendicular to the surface, and one of whose ends dS_1 is just below, and the other dS_2 just above, the surface. The total charge inside this cylinder is σdS. There

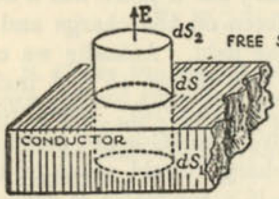


Fig. 8

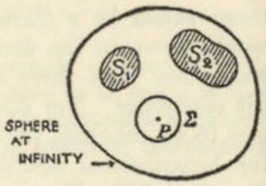


Fig. 9

is no flux of $\mathbf{E}$ across dS_1, or out of the sides of the cylinder, so that the total flux of $\mathbf{E}$ out of volume is $E_n dS_2$. Hence, by Gauss' law $E_n dS_2 = 4\pi\sigma dS$. Now $dS_1 = dS_2 = dS$, so that

$$E_n = 4\pi\sigma \quad . \quad . \quad . \quad . \quad (2)$$

This is **Coulomb's law**; if $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal to the conductor, it may be written

$$\frac{\partial V}{\partial n} = -4\pi\sigma \quad . \quad . \quad . \quad . \quad (3)$$

There is another interesting way in which this result may be obtained. We do this by comparing two different formulæ for V. In § 7 eq. (12) we obtained V by direct summation, and in § 10 eq. (20) we obtained a differential equation for V. Let us now solve this equation and compare the two results.

Consider a series of charged conductors $S_1, S_2, \ldots$ shown shaded in Fig. 9. Let V denote the potential in this system, $V_1, V_2 \ldots$ being the values of V on $S_1, S_2, \ldots$ Any point charges that may be present are to be regarded as very small conductors. Let P be any point outside the conductors; surround P by an extremely small sphere Σ.

In order to solve the equation $\nabla^2 V = -4\pi\rho$, we make use of Green's Theorem.* In its simplest form this states that if $\mathbf{a}$ is any well-behaved field-vector, then

$$\int_{(v)} \operatorname{div} \mathbf{a}\, dv = \int_{(S)} \mathbf{a}.d\mathbf{S} = \int a_n dS, \quad . \quad . \quad (4)$$

where S is any closed surface enclosing a volume v. If we put

$$\mathbf{a} = \phi_1 \operatorname{grad} \phi_2 - \phi_2 \operatorname{grad} \phi_1,$$

where ϕ_1 and ϕ_2 are any two arbitrary scalar functions, this takes the form † more suited to our purposes,

$$\int_{(v)} \left(\phi_1\nabla^2\phi_2 - \phi_2\nabla^2\phi_1\right) dv = \int_{(S)} \left(\phi_1 \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial \phi_1}{\partial n}\right) dS \quad . \quad (5)$$

Now put $\phi_2 = 1/r$, where r is measured from P, and put $\phi_1 = V$, so that $\nabla^2\phi_2 = 0$, and $\nabla^2\phi_1 = -4\pi\rho$. Take the volume v to be all space between $S_1, S_2, \ldots \Sigma$ and the sphere at infinity. The left-hand side of (5) becomes simply $4\pi\int\frac{\rho dv}{r}$. The right-hand side involves integrations over $S_1, S_2, \ldots \Sigma$, and the sphere at infinity in which $\frac{\partial}{\partial n}$ denotes

* Rutherford, *Vector Methods*, 1946, p. 74. But Rutherford uses ds where we use dS.

† Rutherford, p. 75.

differentiation out of v, i.e., out of the sphere at infinity and into the surfaces S_1, S_2.

Since $\phi_1 \to 0$ at large distances, a simple consideration of orders of magnitude shows that the contribution from the sphere at infinity is zero.

To calculate the contribution from Σ we may put $\phi_1 = V_P$ in the first term and treat it as a constant. Also $\dfrac{\partial}{\partial n} = -\dfrac{\partial}{\partial r}$ and $dS = r^2 d\omega$. This gives $\int \left(V_P + r\dfrac{\partial V}{\partial r}\right) d\omega$. If we let the radius of Σ tend to zero this integral becomes $4\pi V_P$.

The contribution from S_1 is obtained by putting $\phi_1 = V_1$, and noting that $\dfrac{\partial \phi_2}{\partial n} = +\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\right)$ cos θ, where θ is the angle between the radius from P and the inward normal to S_1. There are two terms; the first one is

$$V_1 \int_{(S_1)} \frac{-\cos\theta}{r^2} dS = -V_1 \int_{(S_1)} d\omega = 0,$$

since P is outside S_1. The second term is $-\int \dfrac{\partial V}{\partial n}\dfrac{dS}{r}$, in which $\dfrac{\partial}{\partial n}$ denotes differentiation along the outward normal to v, i.e. the inward normal to S_1. This can be written $+\int \dfrac{\partial V}{\partial n}\dfrac{dS}{r}$, where $\dfrac{\partial}{\partial n}$ now denotes differentiation along the outward normal to S_1.

Collecting all the terms together we write (5) in the form

$$V_P = \int \frac{\rho dv}{r} - \frac{1}{4\pi}\int \frac{\partial V}{\partial n}\frac{dS}{r} \quad . \quad . \quad . \quad (6a)$$

The last integral represents integration over the surfaces of all the conductors $S_1, S_2, \ldots$ This formula may be regarded

as the formal solution of Poisson's equation. If we compare it with the known formula

$$V = \int \frac{\rho dv}{r} + \int \frac{\sigma dS}{r} \quad . \quad . \quad . \quad . \quad (6b)$$

we see at once that $\dfrac{\partial V}{\partial n} = -4\pi\sigma$ at the surface of any charged conductor. This is the proof of Coulomb's law (2).

§ 14. Mechanical force on a charged conducting surface

Since there is a field at the surface of a charged conductor the charge on this surface will experience a force; this force, like the field itself, will be everywhere normal to the surface. To calculate it we must consider in rather more detail the way in which the field changes from zero just inside the conductor to $4\pi\sigma$ just outside. Hitherto we have supposed this change to take place discontinuously; in fact, of course, the electric charge on the surface will occupy a very thin layer (see Fig. 10) of width t, which is of the order of a few atomic diameters.

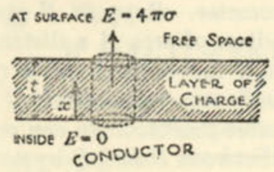


FIG. 10

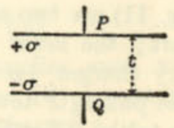


FIG. 11

Suppose that at distance x from the conductor the density of charge is ρ and the field $\mathbf{E}$. Then from Poisson's equation, since $\mathbf{E}$ is normal to the conductor

$$\frac{dE}{dx} = 4\pi\rho \quad . \quad . \quad . \quad . \quad (7)$$

Now consider the charge at different distances inside a small

cylinder of unit cross section shown dotted in Fig. 10. The amount of charge between the planes x, $x+dx$ is ρdx. The force on this charge is

$$E\rho dx = \frac{1}{4\pi} E \frac{dE}{dx} dx, \text{ by (7).}$$

Thus the total force on the charge in this cylinder is

$$\int_0^t \frac{1}{4\pi} E \frac{dE}{dx} dx = \frac{1}{4\pi} \int_0^{4\pi\sigma} EdE = 2\pi\sigma^2 \quad . \quad . \quad (8)$$

Hence the required force is $2\pi\sigma^2$ per unit area of surface of the conductor.

This is a mechanical force acting on the charges. But since the charges cannot usually escape from the conductor, it is communicated by them to the surface itself. If, however, we put too great a charge on the surface, the force on the charges becomes so big that a temporary discharge in the form of a spark does take place from the conductor.

§ 15. Attracted-disc electrometer

The simplest apparatus that makes use of this mechanical force is the attracted-disc electrometer. Suppose P and Q (Fig. 11) are two similar parallel discs of area A a distance t apart; the lower surface of P and the upper surface of Q carry charges $+\sigma$ and $-\sigma$ per unit area respectively. The force pulling P towards Q is therefore $2\pi\sigma^2 A$. We can relate this to the difference in potential between P and Q by noting that the field $E = 4\pi\sigma$, and if the plates are fairly close together the total drop V in potential between P and Q is $E \times t$, so that $V = 4\pi\sigma t$. The attractive force is now $2\pi A(V/4\pi t)^2$, i.e. $V^2 A/8\pi t^2$. If we measure this force by making it pull down one arm of a balance we are provided with an accurate way of determining a given potential difference V.

It is necessary for the plates to be fairly close together, for otherwise on account of edge effects the field is not

uniform across the gap between them. In practice this difficulty is overcome by dividing one of the plates into an inner and outer part, and by measuring the pull on the central part only, where the field may be taken to be uniform. The outer part is called a **guard ring**.

§ 16. Energy of the electrostatic field

Our next problem is to calculate the mutual potential energy between a given set of charges $e_1, e_2 \ldots e_n$ at points $A, B, \ldots$ Let $r_{12}, r_{13}, \ldots$ denote the distances between charges e_1 and e_2, e_1 and e_3, etc. Then the mutual potential energy is clearly $\Sigma \frac{e_i e_j}{r_{ij}}$, where the double summation includes all pairs of distinct values of i and j. We may write this in the form

$$\tfrac{1}{2}e_1\left\{\frac{e_2}{r_{12}} + \frac{e_3}{r_{13}} + \ldots + \frac{e_n}{r_{1n}}\right\} + \tfrac{1}{2}e_2\left\{\frac{e_1}{r_{21}} + \frac{e_3}{r_{23}} + \ldots + \frac{e_n}{r_{2n}}\right\} + \ldots$$

The first expression in brackets is just the potential at A due to all the other charges. If we call this V_1, with a similar meaning for $V_2, \ldots$ then the required mutual potential energy is W, where

$$W = \tfrac{1}{2} e_1 V_1 + \tfrac{1}{2} e_2 V_2 + \ldots = \Sigma \tfrac{1}{2} eV \quad . \quad . \quad (9)$$

An alternative proof of (9) which applies even when there are dielectrics present (see Chapter IV) is obtained by supposing that we build up the final charges $e_1, e_2, \ldots$ stage by stage, always keeping their ratios constant. Thus suppose that at one stage the charges present are $ke_1, ke_2, \ldots ke_n$, where k is some fraction less than one. By the principle of superposition the corresponding potentials are $kV_1, kV_2, \ldots kV_n$. Now let us bring up charges e_1dk, $e_2dk, \ldots e_ndk$. The work that we do is dW, where *

$$dW = \Sigma(e_1dk)(kV_1) = (\Sigma eV)kdk.$$

The total work done W is obtained by integrating from

* In calculating dW here and elsewhere we neglect the self-energy of each charge e.

$k = 0$ to $k = 1$, by which time we have built up the complete charges $e_1, e_2, \ldots e_n$ in their proper positions. Thus

$$W = \int_0^1 (\Sigma eV)kdk = (\Sigma eV)\int_0^1 kdk = \tfrac{1}{2}\,\Sigma eV,$$

which is the same value as in (9).

If the charges are not localised as point charges, but are distributed with volume density ρ and surface density σ the corresponding formula is evidently

$$W = \tfrac{1}{2}\int \rho V dv + \tfrac{1}{2}\int \sigma V dS \quad . \quad . \quad . \ (10)$$

We can rewrite this last formula in a rather different way. For the first term may be transformed as follows:

$$\begin{aligned}\tfrac{1}{2}\int \rho V dv &= \frac{1}{8\pi}\int V \operatorname{div} \mathbf{E}\, dv \\ &= \frac{1}{8\pi}\int \{\operatorname{div}(V\mathbf{E}) - (\mathbf{E}.\operatorname{grad} V)\} dv \\ &= \int \frac{VE_n}{8\pi} dS + \int \frac{E^2}{8\pi} dv.\end{aligned}$$

The surface integral is taken over the sphere at infinity, where a consideration of orders of magnitude shows that it is zero for any finite set of charges; and over the surface of each conductor where, following the notation of Fig. 9, E_n is the normal component of $\mathbf{E}$ away from v into each separate conductor. But by Coulomb's law (2), $E_n = -4\pi\sigma$, so that

$$\tfrac{1}{2}\int \rho V dv + \tfrac{1}{2}\int \sigma V dS = \int \frac{E^2}{8\pi} dv.$$

Hence the total electrostatic energy is

$$W = \int \frac{E^2}{8\pi} dv \quad . \quad . \quad . \quad . \ (11)$$

This integration is to be taken over all space except that occupied by conductors. But since $\mathbf{E} = 0$ inside any conductor, it may equally well be extended over all space.

Equation (11) is interesting because it shows that we may regard the energy of the charges as being distributed throughout all space, the density of energy being $E^2/8\pi$ per unit volume. We sometimes speak of this as the energy density of the field. In the earlier work of Faraday and Maxwell, who could not understand how charges could exert forces on each other across the intervening vacuum, and therefore sought to dispense with the idea of action-at-a-distance, it was supposed that there was an æther pervading all space; this transmitted the inverse square law of attraction from one charge to another by becoming stressed, and it was argued that such a stress would require energy to create it; this was the energy $E^2/8\pi$. Alternatively we may think of the tubes of force as being in a state of tension. Indeed the student will notice a close similarity between the expression $E^2/8\pi$ just obtained, and the mechanical force $2\pi\sigma^2$ per unit area on the surface of a conductor; for since by Coulomb's law, $E = 4\pi\sigma$ at the surface, $2\pi\sigma^2$ is exactly equal to $E^2/8\pi$. We return to this question in § **30**.

More recent theories of the electric field do not make use of the idea of the æther, but it is still convenient to regard the energy of this kind of electrostatic system as in some way residing in the medium with density $E^2/8\pi$.

§ 17. Introducing a new conductor lessens the energy

We next prove that introducing a new uncharged conductor into an electric field lowers the total energy. For suppose the conductor is assembled bit by bit, each element being brought into position uncharged, and electrical communication between the elements being forbidden; as each element is uncharged no work is done. Now allow electrical contact to be established. Positive and negative charges will flow over the surfaces till the equilibrium configuration is reached.

But since charge always flows from regions of higher potential to regions of lower potential, electrical energy is lost. This proves our theorem that introducing an uncharged conductor into a field decreases the energy.

Let us now connect this new conductor to earth (i.e. zero potential). There will, in general, be a further redistribution of charge, and the electrical energy will be further reduced.

§ 18. Dipoles

It frequently happens that we have a pair of equal and opposite charges $\pm e$ at a constant distance l apart: e is usually very large and l is very small, in such a way that the product el has a finite value m. Such a combination is called an **electric dipole**, and m is its **moment**. Most molecules, e.g. HCl and NH_3, are electric dipoles and the theory of such systems is of great importance in physical chemistry. It also turns out that the mathematics used in discussing electric dipoles is identical with that required later in our discussion of permanent magnetism.

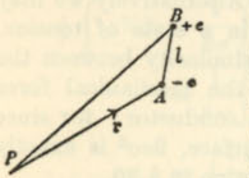


FIG. 12

The moment m is really a vector $\mathbf{m}$, whose direction is from the negative to the positive charge. If we have a dipole $\mathbf{m}$ at AB (Fig. 12) the direction of $\mathbf{m}$ is along the line AB, and the resulting potential at P is

$$V = -\frac{e}{AP} + \frac{e}{BP}$$

$$= e \times \text{change in } 1/r \text{ from } A \text{ to } B$$

$$= el\frac{\partial}{\partial s}\left(\frac{1}{r}\right), \text{ where } \frac{\partial}{\partial s} \text{ denotes differentiation in the direction } AB,$$

$$= \mathbf{m}.\text{grad}\,\frac{1}{r}.$$

In performing the differentiation of $1/r$ we consider the end P of the vector $\mathbf{r}$ as fixed and differentiate with respect to the coordinates of A. For that reason we often write this

$$V = \mathbf{m}.\text{grad}_A\,\frac{1}{r}. \quad . \quad . \quad . \ (12)$$

Since $r = \sqrt{\{(x_A-x_P)^2+(y_A-y_P)^2+(z_A-z_P)^2\}}$, it follows that $\text{grad}_A\frac{1}{r} = -\text{grad}_P\frac{1}{r}$, and so the potential (12) is equally well written

$$V_P = -\mathbf{m}.\text{grad}_P\,\frac{1}{r}. \quad . \quad . \quad . \ (13)$$

If the dipole is at the origin, as in Fig. 13, the potential at P, whose position vector is $\mathbf{r}$, is just

$$V_P = \frac{m\cos\theta}{r^2} = \frac{\mathbf{m}.\mathbf{r}}{r^3}, \quad . \quad . \quad . \ (14)$$

an important formula that should be remembered.

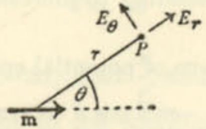


FIG. 13

The electric field at P is most conveniently expressed in terms of the radial and transverse components E_r and E_θ. Thus, since $\mathbf{E} = -\text{grad}\,V$,

$$E_r = -\frac{\partial V}{\partial r} = \frac{2m\cos\theta}{r^3},$$

$$E_\theta = -\frac{\partial V}{r\partial\theta} = \frac{m\sin\theta}{r^3}. \quad . \quad . \quad . \ (15)$$

This shows that the field varies as the inverse cube of the distance, but only when $\theta = 0$ or $\theta = \pi$ is its direction along the radius vector.

§ 19. Forces on dipoles

Suppose that a dipole $\mathbf{m}$ is free to rotate about its centre in the presence of a constant electric field $\mathbf{E}$. Fig. 14 shows a section in the plane through the dipole and the direction of the field. The field acts from left to right in the diagram, and therefore exerts a force $+e\mathbf{E}$ on the charge $+e$, and $-e\mathbf{E}$ on the charge $-e$.

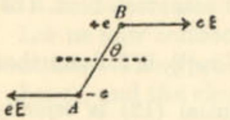


Fig. 14

These are equivalent to a couple of magnitude $eEl \sin\theta$, i.e. $mE\sin\theta$. Since this is about an axis perpendicular to $\mathbf{m}$ and $\mathbf{E}$, it can be written

$$\text{Couple} = \mathbf{m}\times\mathbf{E}, \quad . \quad . \quad . \quad (16)$$

where $\times$ denotes vector product.

If W is the potential energy of the dipole in the field, then

$$-\frac{dW}{d\theta} = \text{couple tending to increase } \theta = -mE\sin\theta.$$

So if we choose the zero of potential energy at $\theta = \pi/2$,

$$W = -mE\cos\theta = -\mathbf{m}.\mathbf{E}\,. \quad . \quad . \quad (17)$$

Alternatively we may argue that if V denotes the potential due to the field $\mathbf{E}$, then

$$\begin{aligned} W &= eV_B - eV_A \\ &= el\frac{\partial V}{\partial s} \qquad \text{(cf. §18)} \\ &= \mathbf{m}.\text{grad } V \\ &= -\mathbf{m}.\mathbf{E}. \end{aligned}$$

Incidentally if the direction of the field is along the positive direction of the z axis, the potential V used above has the form, which we shall often use later

$$V = -Ez + \text{constant}.$$

Our next problem is to calculate the mutual potential energy of two given dipoles. We can use the same method that gave us the potential due to one dipole in equations (12) and (13). Let O_1 and O_2 be the centres of the two dipoles whose moments are $\mathbf{m}_1$ and $\mathbf{m}_2$. Then if V denotes the potential due to $\mathbf{m}_1$ and if we suppose that $\mathbf{m}_2$ arises from charges $\pm e_2$ at distance $\mathbf{l}_2$, as in Fig. 15, the mutual potential energy is

$$\begin{aligned} -e_2V_A + e_2V_B &= e_2\mathbf{l}_2 \,.\, \text{grad } V \\ &= \mathbf{m}_2 \,.\, \text{grad } V. \end{aligned}$$

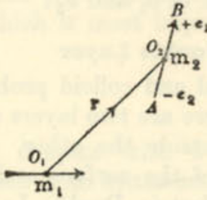


Fig. 15

But from (13), $V = -\mathbf{m}_1.\text{grad}\,\frac{1}{r}$, and both differentiations are made with respect to the coordinates of O_2. Thus the mutual potential energy is W, where

$$W = -\mathbf{m}_2.\text{grad}\left(\mathbf{m}_1.\text{grad}\,\frac{1}{r}\right) = \mathbf{m}_2.\text{grad}_{O_2}\left(\frac{\mathbf{m}_1.\mathbf{r}}{r^3}\right),$$

and $\mathbf{r}$ stands for the vector O_1O_2. By working in terms of Cartesian coordinates, or by vector differentiation the reader will have no difficulty in verifying that this may be written

$$W = \frac{\mathbf{m}_1.\mathbf{m}_2}{r^3} - \frac{3(\mathbf{m}_1.\mathbf{r})(\mathbf{m}_2.\mathbf{r})}{r^5} \quad . \quad . \quad (18)$$

A convenient form for (18) may be obtained by calling ϵ the angle between the dipoles, and θ_1, θ_2 the two angles between

the dipoles and the line of centres O_1O_2. Then the mutual energy is

$$W = \frac{m_1 m_2}{r^3}\{\cos\epsilon - 3\cos\theta_1\cos\theta_2\}\ . \qquad . \ (19)$$

It is an easy matter to calculate from (19) the force and couple exerted on $\mathbf{m}_2$ by $\mathbf{m}_1$. Thus the force along the direction O_1O_2 is $-\dfrac{\partial W}{\partial r}$, and the couple tending to increase θ_2 is $-\dfrac{\partial W}{\partial \theta_2}$. In this latter differentiation, however, ϵ must be regarded as a function of θ_1 and θ_2.

§ 20. The Electric Double Layer

In many biological and colloid problems it appears that on a given surface there are two layers of charge, of opposite sign, the one just outside the other. These may cover a part, or the whole, of the surface and together they form what is called an Electric Double Layer. The **strength** p of the layer is the product of the charge per unit area and the distance apart of the two layers. Thus (see Fig. 16), if σ is the surface density of charge, then $p = \sigma t$. The two charges $\pm\sigma dS$ shown in the figure are together equivalent to a dipole of moment $\sigma t dS$, i.e. pdS. Thus instead of speaking of layers of positive and negative charge we may equally well speak of a layer of dipoles, all pointing normally to the surface, rather like the bristles of a hair-brush, such that the dipole moment per unit area is p. In Chapter VI we shall find this interpretation extremely useful.

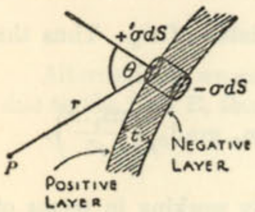


FIG. 16

It is an easy matter to calculate the potential at P due to an electric double layer of strength p. For since the element dS in Fig. 16 behaves like a dipole of moment pd**S** its contribution to the potential at P is given by (14); it is $pdS\cos\theta/r^2$, where θ is the angle between the direction of the normal and the direction to P. Now $dS\cos\theta/r^2 = d\omega$, so that the potential at P is

$$V = \int p\,d\omega\ . \quad . \quad . \quad . \ (20)$$

If the double layer is of uniform strength, this becomes

$$V = p\omega, \quad . \quad . \quad . \quad . \ (21)$$

where ω is the solid angle subtended at P by the boundary curve of the double layer. There is a difference of potential $4\pi p$ between points just on either side of the double layer. It is this difference which is most important in applications of this paragraph.

§ 21. Condensers

We conclude this chapter with a brief discussion of condensers. A condenser is an instrument for storing charge; it consists of two **plates** which are conductors of arbitrary shape. The positive plate carries a charge $+Q$ and the negative plate a charge $-Q$, so that there is no net charge on the condenser as a whole. If the potential difference between the plates is V, we define the **capacity** C by the formula

$$C = Q/V, \text{ i.e. } Q = VC\ . \quad . \quad . \ (22)$$

By the principle of superposition, if we increase Q in a certain ratio, V is also increased in the same ratio, so that C is a constant independent of the charge on the condenser plates. In fact C is a purely geometrical constant depending on the position and shape of the two plates. To calculate C for a given condenser, we must first solve the potential equation; this is best illustrated by a few simple examples.

Parallel Plate Condenser.—Consider two equal plates of area A a distance t apart, as in Fig. 17. If σ is the surface density of charge on the positive plate M, $-\sigma$ will be the corresponding density on the negative plate N. Apart from a small edge-correction the tubes of force will go straight

from M to N. It follows that the field between the plates is directed from M to N and since, from § 10, equation 19, div $\mathbf{E} = 0$, this involves $\mathbf{E} =$ constant. By Coulomb's law this constant is $4\pi\sigma$. The total drop of potential between M and N is therefore $V = 4\pi\sigma t$. But the total charge Q is simply

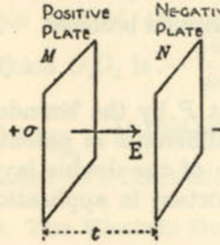


Fig. 17

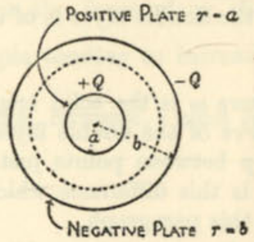


Fig. 18

σA. Hence the capacity is $C = Q/V = A/4\pi t$. Evidently this is a purely geometrical quantity independent of the charge σ and having the dimensions of a length. The largest value of C is obtained when t is small; for this reason the plates are usually close together.

Two Concentric Spheres (Fig. 18).—By symmetry the field is everywhere radially outwards. So by applying Gauss' flux theorem to a sphere of radius r shown dotted, we have

$$E \times 4\pi r^2 = 4\pi Q,$$

where Q is the total charge on the inner conductor $r = a$. Hence

$$-\frac{dV}{dr} = E = \frac{Q}{r^2}.$$

So

$$V = \text{constant} + \frac{Q}{r}.$$

The difference in potential $V_a - V_b$ between the plates is $Q\left(\frac{1}{a} - \frac{1}{b}\right)$. Therefore the capacity is

$$C = Q/(V_a - V_b) = ab/(b-a) \qquad . \quad . \ (23)$$

If we make b tend to infinity we have a condenser one of whose plates is a sphere of radius a, and the other is at infinity. From (23)

$$C = \lim_{b\to\infty} \frac{ab}{b-a} = a. \qquad . \quad . \quad . \quad . \ (24)$$

Condensers in Series.—If a set of n condensers are joined together to form a single condenser in such a way that the negative plate B_1 of one is joined to the positive plate A_2 of the next, and so on, the charges on the various plates, when $\pm Q$ are placed on A_1 and B_n, must be $+Q$, $-Q$, $+Q$, $-Q$, . . . as in the diagram (Fig. 19). The total drop of

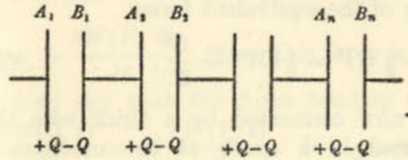


Fig. 19

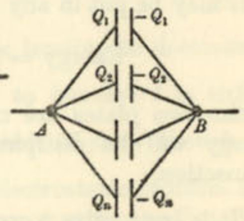


Fig. 20

potential V from A_1 to B_n is the sum of the individual differences of potential across the separate condensers. Thus

$$V = Q/C_1 + Q/C_2 + \ . \ . \ . \ + Q/C_n.$$

But when A_1 and B_n are regarded as the two plates of a complete condenser, the total charge stored is still only Q, so the capacity is C, where

$$1/C = V/Q = 1/C_1 + 1/C_2 + \ . \ . \ . \ + 1/C_n\,. \qquad . \ (25)$$

This is described as joining the condensers **in series** or **in cascade.**

Condensers in Parallel.—An alternative way of forming one condenser out of several is to join all the positive plates together at A, and all the negative plates at B, as in Fig. 20. The potential across each separate condenser is now V, but

the charges on each are different, being given by the equations $Q_1 = C_1V$, etc. The total charge on the positive plate is $Q=Q_1+Q_2+ \ldots +Q_n = V(C_1+C_2+ \ldots +C_n)$. Thus the capacity C of the system regarded as one condenser, is

$$C = Q/V = C_1+C_2+ \ldots +C_n \quad . \quad . (26)$$

This is described as joining the condensers **in parallel.**

A condenser whose plates carry charges $\pm Q$ at potentials V_1 and V_2 will be a reservoir of electrical energy. In fact, from (9)

$$\text{Energy} = \Sigma \tfrac{1}{2} eV = \tfrac{1}{2}(QV_1+(-Q)V_2) = \tfrac{1}{2}QV,$$

where V is the potential difference $V_1 - V_2$ across the plates. This may be put in any of the equivalent forms

$$\text{Energy} = \tfrac{1}{2}QV = \tfrac{1}{2}CV^2 = \frac{Q^2}{2C}. \quad . \quad . (27)$$

If the two plates are now connected by a thick wire this energy will be dissipated in a spark at the moment of connection.

It is instructive to compare (27) with the formula $\int \frac{E^2}{8\pi} dv$ for the energy, which was proved in (11). Consider, for example, the parallel plate condenser of Fig. 17. We have seen that E has the value $4\pi\sigma$ everywhere between the plates, and effectively zero elsewhere. Hence the energy integral is

$$\int \frac{E^2}{8\pi} dv = \frac{(4\pi\sigma)^2}{8\pi} \times \text{volume of condenser}$$
$$= 2\pi\sigma^2 At.$$

But $\sigma A = Q$, and $4\pi\sigma t = V$, so that this formula may be written

$$\text{Energy} = \tfrac{1}{2}QV, \text{ in agreement with (27).}$$

This is one of the few cases where it is possible to verify by simple direct integration that $\int \frac{E^2}{8\pi} dv$ does give the same value for the energy as $\Sigma \frac{1}{2} eV$.

§ 22. Mechanical forces on a conductor

Consider again the parallel plate condenser of Fig. 17. The energy W may be regarded as a function either of the charge Q and distance t, or of the potential V and distance t. Calling these functions $W(Q, t)$ and $W(V, t)$ respectively, we have, from (27):

$$W(Q, t) = Q^2/2C = 2\pi Q^2 t/A,$$
$$W(V, t) = CV^2/2 = AV^2/8\pi t.$$

Now according to (8) the mechanical force on the negative plate is $2\pi\sigma^2 A$, which may be written either as $+\dfrac{\partial W(Q, t)}{\partial t}$ or as $-\dfrac{\partial W(V, t)}{\partial t}$. Since this force is tending to decrease t, we may say that the force tending to increase t is either $-\dfrac{\partial W(Q, t)}{\partial t}$ or $+\dfrac{\partial W(V, t)}{\partial t}$. It is possible to show, though we shall not do so here,* that if in any electrostatic problem the energy W can be calculated in the above manner as a function of the charges or the potentials, together with a coordinate t which defines the position of one of the conductors, the mechanical force tending to increase t is always given by the two above forms. Our previous example is a particular case of this theorem, in which t is the distance between the two plates of the condenser.

§ 23. Examples

1. Show that with a system of charged conductors in which the total net charge is zero, at least one conductor must be everywhere charged positively, and one everywhere negatively.

2. S is a surface completely surrounding a given system of charges, and P is a point outside S. Show by reasoning similar to that of § 13 that the potential at P is the same as would be obtained if the system of charges was replaced by a surface

* See e.g. Jeans *Electricity and Magnetism*, Chapter IV.

distribution $-\dfrac{1}{4\pi}\dfrac{\partial V}{\partial n}$, and an electric double layer of strength $\dfrac{V}{4\pi}$ per unit area, both on the surface S. (This is known as **Green's Equivalent Stratum**).

3. Point charges $e_1, e_2 \ldots$ at $A, B, \ldots$ are such that the potential at A due to all the charges except e_1 is V_1, and at B due to all except e_2 is V_2, and so on. A second set of charges $e_1', e_2' \ldots$ at $A, B, \ldots$ gives potentials $V_1', V_2' \ldots$ Prove **Green's Reciprocal Theorem** which states that

$$e_1V_1'+e_2V_2'+\ldots = e_1'V_1+e_2'V_2+\ldots$$

Show that the same result is true if $A, B, \ldots$ are a series of conductors on which the total charges are $e_1, e_2 \ldots$, and whose potentials are $V_1, V_2, \ldots$

Now make all the charges zero except e_1 and e_2', and reduce the second conductor to a point P. Deduce the following theorem: the potential of an uncharged conductor under the influence of a unit charge at P is the same as the potential at P due to a unit charge placed on the conductor. Hence show that a unit charge placed a distance f away from the centre of an uncharged conducting sphere raises the latter to potential $1/f$.

4. A spherical soap bubble is blown on the end of a thin tube, and is then given an electric charge Q. The tube is now made open to the air, so that the air pressure is the same inside and outside. It is known that if the radius is r, the effect of surface tension is to give an inward pressure $2T/r$ per unit area. If this is balanced by the mechanical force due to the charge, so that the bubble is in equilibrium, show that $Q^2 = 16\pi Tr^3$.

5. S is a closed surface drawn in such a way that it encloses no electric charges. Show that according to Maxwell's theory the energy stored in that part of the field lying inside S may be written

$$\frac{1}{8\pi}\int V\frac{\partial V}{\partial n}\,dS,$$

the integration being over the surface of S.

6. Two condensers of capacities C_1 and C_2 carry charges Q_1 and Q_2. Their negative plates are now connected to earth and their positive plates are joined. Show that the final potential of the positive plates is $(Q_1+Q_2)/(C_1+C_2)$, and that there is a loss of energy equal to $(Q_1C_2-Q_2C_1)^2/2C_1C_2(C_1+C_2)$. What happens to the energy?

7. Verify the theorem in § 16 equation (11) for the case of a conducting sphere of radius a which receives a charge Q.

8. A conducting sphere of radius a receives a charge Q. Verify by direct integration of equation (6b) that the potential is Q/r at points outside the sphere, and Q/a at points inside. Determine the electric field E at any point, and by performing the integration $\int\dfrac{E^2}{8\pi}\,dv$ verify that the energy of the condenser formed by the given sphere and the sphere at infinity is $Q^2/2a$. Notice that this implies that a genuine point charge $(a\to 0)$ would have infinite self-energy.

9. Three concentric hollow conducting spheres have radii a, b, c. The inner and outer spheres are connected together by a fine wire, and form one plate of a condenser; the middle sphere is the other plate. By regarding this system as two separate condensers in parallel, or otherwise, show that the capacity is $b^2(c-a)/(b-a)(c-b)$.

10. A condenser consists of two coaxial cylinders of radii a and b, each of length l. Show that its capacity is $l/2\log(b/a)$.

11. S_1 is a fixed hollow conducting cylinder of radius b, and S_2 is a smaller solid conducting cylinder of radius a coaxial with S_1. S_2 is free to slide along its axis and a constant potential difference V is maintained between the two cylinders. Show that when a length l of S_2 is inside S_1, the electrostatic energy is $lV^2/4\log(b/a)$, and hence deduce that the movable cylinder experiences a force $V^2/4\log(b/a)$ drawing it inside the fixed cylinder.

12. Obtain the equations (15) for the field due to a dipole by superposing the fields due to the two component charges, without calculating the potential as an intermediary. Notice how much easier it is to work in terms of the potential rather than the field.

13. A dipole of moment $\mathbf{m}$ is placed in an inhomogeneous field $\mathbf{E}$. Show that in addition to the couple of moment $\mathbf{m}\times\mathbf{E}$ it experiences a resultant force $(\mathbf{m}.\text{grad})\,\mathbf{E}$, where $\mathbf{m}.\text{grad}$ denotes the operator

$$m_x\frac{\partial}{\partial x}+m_y\frac{\partial}{\partial y}+m_z\frac{\partial}{\partial z}.$$

14. An electric dipole of moment m is free to rotate about its centre which is fixed, in the presence of a constant electric field E. If its moment of inertia about the centre is I, and if we may assume that the only force on the doublet is the electrostatic couple due to the field, prove that small oscillations about its equilibrium position are Simple Harmonic, with period $2\pi\sqrt{(I/mE)}$.

15. Two equal charges e are at opposite corners of a square of side a, and an electric dipole of moment m is at a third corner pointing towards one of the charges. If $m = 2\sqrt{2}\, ea$, show that the field strength at the fourth corner of the square is $\sqrt{\frac{17}{2}} \cdot \frac{e}{a^2}$.

16. A and B are two electric dipoles such that the direction of A passes through B, and the direction of B is perpendicular to that of A. Show that the actual force exerted by A on B is not in the same direction as the actual force exerted by B on A. Explain why this does not contradict Newton's third law of motion, that action and reaction are equal and opposite.

17. An electric dipole is at the origin and its direction is that of the polar axis. Show that the lines of force are given by $\sin^2\theta/r =$ constant. Sketch the lines of force and the equipotentials which lie in a given plane through the dipole.

18. A system of charges consists of $+2e$ at the origin and $-e$ at the two points $(\pm a, 0, 0)$. Show that at distances from the origin much greater than a, the potential may be written in the approximate form

$$V = -\frac{ea^2}{r^3}(3\cos^2\theta - 1).$$

This is sometimes called a **quadrupole**, and is important in studying the forces between atoms and molecules.

19. An electric double layer of uniform strength p is distributed over the surface of a sphere. Show that the potential is zero at all external points, but has the value $4\pi p$ at all internal points.

20. Show that on crossing a double layer of strength p the potential changes by $4\pi p$. Illustrate this by considering in detail the variation of potential across the thickness t of the layer in Fig. 16.

21. An electric double layer of uniform strength p is spread over one side of a circular disc of radius a. Show that at points distant x from the layer along the line of symmetry perpendicular to the plane of the disc, the electric field is $2\pi pa^2/(a^2+x^2)^{3/2}$.

22. If the law of force between charges e_1 and e_2 was e_1e_2/r^n, show that the potential due to a charge e would be $e/(n-1)r^{n-1}$, and the potential due to an electric dipole of moment $\mathbf{m}$ would be

$$(\mathbf{m}\,.\,\mathbf{r})/r^{n+1}.$$

CHAPTER IV

DIELECTRICS

§ 24. Dielectric constant

We have hitherto supposed that our experiments were all carried out in free space, i.e. a vacuum. We must now discuss the result of placing other materials in the electric field. It was Faraday who discovered that if the space between the two plates of a condenser was filled with an insulating material such as glass or mica, its capacity was multiplied by a certain constant K, which depended on the material but was independent of the shape of the condenser. Such substances are called **dielectrics**, and K is the **dielectric constant**, or **specific inductive capacity.** Values of K range from 1 for free space and 6 for glass to about 81 for water. For air at normal pressure K is 1·0006, which shows that the results of Chapters II and III are almost unaffected if we use air instead of a vacuum.

§ 25. Polarisation

In order to explain this phenomenon we must consider in more detail what happens when an electric field $\mathbf{E}$ falls upon dielectric material. As we saw in Chapter I, each atom of the material consists of positive and negative charges, and so, when the field is applied, the positive charges will be pulled in one direction and the negative charges in the opposite direction. If there is no resultant charge on the atom there will be no net force on it either, so that the effect of the field will be to separate slightly the positive from the negative charges. We say that the medium is polarised

by the field. As a result of this polarisation each atom or molecule becomes a tiny dipole whose strength will depend upon $\mathbf{E}$. If the field is not too large this strength is proportional to $\mathbf{E}$ and we may write it $\alpha\mathbf{E}$, where α is the **polarisability**. We call $\alpha\mathbf{E}$ the induced dipole; for an isolated atom, and indeed for nearly all substances it is parallel to $\mathbf{E}$; however, certain crystals and other anisotropic media are exceptions to this rule.

There is also another type of contribution to the polarisability α. This occurs when the molecules of the dielectric are themselves permanent dipoles, as e.g. water. In the absence of a field these permanent dipoles will point equally often in all directions, but when a field is applied, there is a couple (see § 19) tending to orient them relative to the field, and there will consequently be a resultant moment in the direction of the field. Let us suppose that this effect also has been included in the value of α. Then, if there are N molecules in unit volume, the total moment induced by an electric field $\mathbf{E}$ is $N\alpha\mathbf{E}$ per unit volume.* This is referred to as the **polarisation**. It is a vector quantity and we shall label it $\mathbf{P}$. In isotropic materials

$$\mathbf{P} = N\alpha\mathbf{E} = k\mathbf{E}, \qquad . \quad . \quad . \quad (1)$$

where $k = N\alpha$. With anisotropic materials there is still a polarisation $\mathbf{P}$, but it is not necessarily related to $\mathbf{E}$ according to (1). The constant k is called the **dielectric susceptibility**; it depends upon the density of the material, as given by N, and upon the polarisability α. Both of these may depend upon the temperature.

Thus the effect of an electric field $\mathbf{E}$ upon a dielectric is to create the polarisation $\mathbf{P}$, with its associated dipole moment $\mathbf{P}\,dv$ in each volume element dv. In calculating the potential for such a system we may now forget completely about the

* There is a small difficulty here, for the dipoles in the immediate neighbourhood of any one dipole will themselves contribute to the field which polarises the original dipole. Except when the polarisation is very large the difficulty may be overcome by a suitable change in the definition of α.

material of the dielectric itself, provided that we imagine it replaced by the volume polarisation $\mathbf{P}\,dv$.

Now from § **18**, eq. 12, we know that the potential at any point due to a single dipole $\mathbf{m}$ is $\mathbf{m}.\mathrm{grad}\,\frac{1}{r}$, the differentiation being taken with respect to the coordinates of the dipole. Putting $\mathbf{m} = \mathbf{P}\,dv$, and integrating, we see that the potential arising from the induced polarisation may be written $\int \mathbf{P}.\mathrm{grad}\,\frac{1}{r}\,dv$. But

$$\mathrm{div}\,\frac{\mathbf{P}}{r} = \frac{1}{r}\,\mathrm{div}\,\mathbf{P} + \mathbf{P}.\mathrm{grad}\,\frac{1}{r},$$

so that this integral is

$$\int \mathrm{div}\,\frac{\mathbf{P}}{r}\,dv - \int \frac{1}{r}\,\mathrm{div}\,\mathbf{P}\,dv.$$

By Green's theorem we can transform this to

$$\int \frac{\mathbf{P}.d\mathbf{S}}{r} - \int \frac{1}{r}\,\mathrm{div}\,\mathbf{P}\,dv \quad . \quad . \quad . \quad (2)$$

The surface integral is taken over the boundary of the dielectric.

This last equation is important. For it shows that when we have a polarisation $\mathbf{P}$, the resulting potential is exactly the same as we should obtain if we were to suppose that there was a volume distribution of charge $-\mathrm{div}\,\mathbf{P}$ throughout the dielectric, and a surface distribution P_n on the boundary of the dielectric. These charges are often referred to as **apparent charges** *; on removing the field $\mathbf{E}$, the polarisation disappears and with it the apparent charges.

§ 26. Electric displacement

We are now in a position to formulate the laws of the electrostatic field in the presence of dielectrics. Suppose

* Sometimes also as fictitious charges, or bound charges (Faraday), but although we shall sometimes use the latter, the title is not a happy one.

that at any point in the medium the field is $\mathbf{E}$, with resulting polarisation $\mathbf{P}$. Then in addition to any true charges ρ and σ that may be present, we must include contributions to the potential from the apparent charges $-\mathrm{div}\,\mathbf{P}$ and P_n which take account of the polarisation. Gauss' flux law (§ **10**, eq. 19) now takes the form

$$\mathrm{div}\,\mathbf{E} = 4\pi\,(\rho - \mathrm{div}\,\mathbf{P}),$$

i.e.

$$\mathrm{div}\,(\mathbf{E} + 4\pi\mathbf{P}) = 4\pi\rho.$$

Let us call this vector quantity on the left-hand-side $\mathbf{D}$, so that

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\mathrm{div}\,\mathbf{D} = 4\pi\rho. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The vector $\mathbf{D}$ is extremely important. Maxwell called it the **electric displacement.** As we shall see that it plays a crucial part in all that follows it will be convenient to make a list of the properties of $\mathbf{D}$.

(i) In isotropic materials for which (1) holds, we may write

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = (1 + 4\pi k)\mathbf{E}.$$

If we put

$$1 + 4\pi k = K, \quad . \quad . \quad . \quad . \quad (5)$$

then

$$\mathbf{D} = K\mathbf{E} \quad . \quad . \quad . \quad . \quad (6)$$

We shall shortly identify K with the dielectric constant. Indeed K is often defined by means of (6), and then (5) is known as **Faraday's equation.** But it is important to realise that (6) is only correct if the medium is isotropic; we ought not therefore to define $\mathbf{D}$ by (6) but by (3), which is always true. In free space k has the value 0, K has the value 1, and

$$\mathbf{D} = \mathbf{E} \quad . \quad . \quad . \quad . \quad (7)$$

(ii) Equation (4) may be written as a flux theorem in the form

flux of $\mathbf{D}$ out of any surface $= 4\pi \times$ true included charge . (8)

We may regard this as replacing the earlier flux theorem; it shows that **D** rather than **E** plays the really fundamental role in dielectric media.

(iii) Equation (8) is exactly the counterpart of the flux theorem in Chapter II, eq. (18). We may therefore write down at once several other properties of **D**. For example, there are field lines of **D**, given by the differential equations

$$\frac{dx}{D_x} = \frac{dy}{D_y} = \frac{dz}{D_z} \quad . \quad . \quad . \quad . \quad (9)$$

(iv) Further there are tubes of **D**, and from (4) and (8) these tubes can only begin and end on true charges. There are 4π unit tubes of **D** which start on each unit charge.

(v) Just as in Chapter III, $\mathbf{D} = 0$ in a conductor, and by applying the new flux theorem (8) in a manner similar to § **13**, we obtain the modified Coulomb's law at the surface of a conductor:

$$D_n = 4\pi\sigma, \quad . \quad . \quad . \quad . \quad (10)$$
$$E_n = 4\pi\sigma/K \quad . \quad . \quad . \quad . \quad (11)$$

The student will have no difficulty in adapting the proof of § **14** to show that the mechanical force on a conductor is

$$2\pi\sigma^2/K \text{ per unit area} \quad . \quad . \quad . \quad (12)$$

(vi) Equation (11) enables us to complete the identification of K. For if we have two exactly similar condensers except that one is filled with a medium of uniform dielectric constant K, and the other is in free space; and if we charge them so that the plates have the same charge density σ in the two cases, then the field in the first is $1/K$ times the field in the second. Hence the potential difference is $1/K$ times as great. But the capacity C is merely charge divided by potential difference. So the capacity of the dielectric condenser is K times the capacity of the vacuum condenser. This identifies K as defined by (5) with the dielectric constant introduced by Faraday.

(vii) Next suppose that we have a single charge e in a medium K, and let us apply the flux theorem (8) to a concentric sphere of radius r. By symmetry **D** has the same value at all points of the sphere, and is directed radially outwards. Hence $4\pi r^2 D = 4\pi e$. So $D = e/r^2$, $E = e/Kr^2$, and the potential $V = e/Kr$. It follows that two charges e_1, e_2 at a distance r in a dielectric repel each other with a force

$$\frac{e_1 e_2}{Kr^2} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Thus the inverse square law still holds, but the force is reduced in the ratio $K:1$. The subject of dielectric media is often introduced by means of (13). But we have preferred not to do so for two reasons. First, it is not easy to see just what is meant by force in the case of solid media (see § **31**); and second, the formula (13) applies only if (*a*) K is constant, (*b*) the dielectric is isotropic and (*c*) it also occupies the whole of space. For these reasons it is more fundamental to base everything on the two equations (3) and (4) in which none of these restrictions occur.

(viii) Since for an isotropic medium $\mathbf{D} = K\mathbf{E}$, and $\mathbf{E} = -\text{grad } V$, it follows that $\mathbf{D} = -K \text{ grad } V$. Further $\text{div } \mathbf{D} = 4\pi\rho$, so that Poisson's equation (Chapter II, eq. 20) takes the form

$$\text{div } (K \text{ grad } V) = -4\pi\rho \quad . \quad . \quad . \quad (14)$$

If K is constant in any region this simplifies to

$$\nabla^2 V = -4\pi\rho/K \quad . \quad . \quad . \quad . \quad (15)$$

It is important to realise that unless K is constant, there is no simple integral formula for V corresponding to Chapter II, eq. (12). In certain simple cases it may be possible to solve (15) directly, but unless the shape of the dielectric is particularly simple we have to resort to special methods (see Chapters IX and X), or else numerical approximation.

(ix) Before we can solve equations (14) or (15) we must know what happens at the boundary of a dielectric. Fig. 21 shows the boundary between a dielectric and free space.

Suppose that we take a unit charge round the path $ABCDA$ shown dotted. As the electric field is conservative no net work is done. But if AD and BC are small enough we can neglect the work done along them; it follows that the work

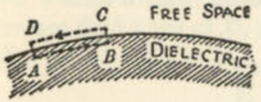


FIG. 21

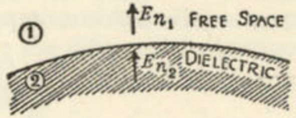


FIG. 22

along AB is equal to the work along DC. This is conveniently expressed by saying that if E_s is the component of **E** in any direction lying in the common boundary, then at a change of medium

$$E_s \text{ is continuous} \quad . \quad . \quad . \quad (16)$$

The boundary condition for the normal component E_n may most easily be obtained by noting (Chapter II, question 12) that when we cross a surface layer of charge σ the normal component of **E** changes by $4\pi\sigma$. Now we saw at the end of § 25 that if the polarisation is **P** there is an effective surface layer of apparent, or bound, charges equal to P_n. Thus E_n changes by $4\pi P_n$, and in the notation of Fig. 22

$$E_{n1} - E_{n2} = 4\pi P_n,$$

i.e.

$$E_{n1} = E_{n2} + 4\pi P_n = D_{n2}.$$

Since **E** and **D** are the same in free space, this implies that across the boundary

$$D_n \text{ is continuous.} \quad . \quad . \quad . \quad . \quad (17)$$

An alternative proof may be obtained by applying the modified Gauss flux theorem (8) to a small cylinder in a manner similar to that of § 13. We leave it as an exercise to the reader to verify that (16) and (17) are equally valid on passing from

one dielectric to another, so that the boundary conditions are:

$$E_{s1} = E_{s2}; \ D_{n1} = D_{n2}, \text{ or } K_1 E_{n1} = K_2 E_{n2} \quad . \quad . \quad (18)$$

§ 27. Parallel plate condenser

It is instructive to consider the particular example of a parallel plate condenser between the plates of which there is placed a slab of dielectric K. If we use the notation of Fig. 23 we shall distinguish three regions: in each region

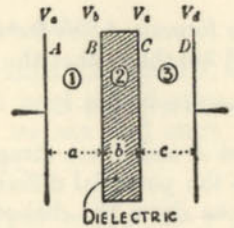


FIG. 23

both **E** and **D** are constant and are directed from one plate to the other. Let the potentials at A, B, C, D be V_a, V_b, V_c, V_d respectively. Then from the fundamental relation $\mathbf{E} = -\text{grad } V$, it follows that

$$E_1 = \frac{V_a - V_b}{a}, \ E_2 = \frac{V_b - V_c}{b}, \ E_3 = \frac{V_c - V_d}{c}. \quad . \quad (19)$$

Regions 1 and 3 are in free space and region 2 has dielectric constant K, so

$$D_1 = E_1, \ D_2 = KE_2, \ D_3 = E_3 \quad . \quad . \quad (20)$$

From the boundary conditions (18) it follows that

$$D_1 = D_2 = D_3 = \beta, \text{ say.}$$

So from (20)

$$E_1 = E_3 = \beta, \ E_2 = \beta/K.$$

Using Coulomb's law (10) we see that the charge density on

plate A is $\beta/4\pi$, so that if A denotes the area of the plates A and D, the condenser carries a charge $\beta\mathsf{A}/4\pi$. The total potential difference, from (19), is

$$V_a - V_d = aE_1 + bE_2 + cE_3 = \beta\left(a + \frac{b}{K} + c\right).$$

Hence the capacity of the condenser is

$$C = \frac{\mathsf{A}}{4\pi\left(a + \frac{b}{K} + c\right)}.$$

Comparison with the formula $\mathsf{A}/4\pi t$ for a condenser of thickness t in free space (§ 21) shows that the equivalent thickness of the given condenser is $a + \frac{b}{K} + c$.

Next suppose that A and D are attached to the terminals of a battery so that the potential difference $V_a - V_d$ is kept constant. Now let us draw the dielectric slab away from the condenser. When the slab is wholly or partly withdrawn, the capacity C is reduced, and hence the charge on the plates, which is $C(V_a - V_d)$, is also reduced. This means that while the slab is being removed, a current will flow from plate A to plate D through the battery. Indeed this can be measured, if we have a sufficiently sensitive instrument.

This condenser illustrates also the theorem on apparent charges at the end of § 25. For inside the dielectric slab the polarisation P is given by

$$E_2 + 4\pi P = D_2 = \beta.$$

So

$$4\pi P = \beta\left(1 - \frac{1}{K}\right).$$

Now $\mathbf{P}$ is constant so that div $\mathbf{P} = 0$, and there is no apparent volume distribution of charge. But P_n at surfaces B and C has the value $\mp \frac{\beta}{4\pi}\left(\frac{K-1}{K}\right)$ respectively. Thus according to

(2) we may replace the dielectric slab by a layer of charge $-\frac{\beta}{4\pi}\left(\frac{K-1}{K}\right)$ per unit area on the surface B and an equal positive layer on the surface C; we may then treat the problem as though there was no dielectric present at all. These two layers of charge would then be responsible for changing E_1 to E_2, and E_2 to E_3 as we pass into and out of the dielectric, as can soon be verified by the use of Gauss' flux theorem.

§ 28. Energy of the field

The previous formula (Chapter III, eq. 11) for the energy of the field W will need modification when dielectrics are present. However we can still start with the formula in Chapter III, eq. (10):

$$W = \Sigma \tfrac{1}{2} eV = \tfrac{1}{2}\int \rho V\, dv + \tfrac{1}{2}\int \sigma V\, dS,$$

since it is easily seen that the second proof of this result in § 16 is unaffected by the presence of dielectrics. We must now substitute div $\mathbf{D} = 4\pi\rho$, and $D_n = 4\pi\sigma$. With reasoning exactly parallel to that of § 16, we write

$$\begin{aligned}\frac{1}{2}\int \rho V\, dv &= \frac{1}{8\pi}\int V \operatorname{div} \mathbf{D}\, dv \\ &= \frac{1}{8\pi}\int \{\operatorname{div} V\mathbf{D} - \mathbf{D}.\operatorname{grad} V\}\, dv \\ &= \frac{1}{8\pi}\int V\mathbf{D}.d\mathbf{S} + \frac{1}{8\pi}\int \mathbf{D}.\mathbf{E}\, dv.\end{aligned}$$

The surface integral is taken outwards over the sphere at infinity—where a consideration of orders of magnitude shows that it is zero—and over the surface of each conductor, D_n being measured into the conductor. Now V is constant on

each conductor, and the inward normal component of $\mathbf{D}$ is $-4\pi\sigma$. We may therefore write the equation in the form

$$\tfrac{1}{2}\int \rho V\,dv + \tfrac{1}{2}\int \sigma V\,dS = \int \frac{\mathbf{D}.\mathbf{E}}{8\pi}\,dv$$

i.e.

$$W = \int \frac{\mathbf{D}.\mathbf{E}}{8\pi}dv. \quad . \quad . \quad . \quad (21)$$

Since $\mathbf{E}$ vanishes inside a conductor, the integration in (21) may be taken over the whole of space, including the conductors.

According to this the energy is distributed over all space with density $\dfrac{\mathbf{D}.\mathbf{E}}{8\pi}$. With isotropic bodies we may write this $\dfrac{KE^2}{8\pi}$ or $\dfrac{D^2}{8\pi K}$. If we are calculating the force on any conductor using the formulæ of § 22 this is the appropriate expression to use for the energy.*

In the case of a crystalline dielectric the relation $\mathbf{D} = K\mathbf{E}$ no longer holds. Taking the directions of x, y, z along the three crystal axes, K has a different value in each direction; we replace the one equation $\mathbf{D} = K\mathbf{E}$ by the three equations

$$D_x = K_1E_x,\ D_y = K_2E_y,\ D_z = K_3E_z \quad . \quad (22)$$

If $K_1 = K_2 = K_3$ the crystal is isotropic; if two of the K's are the same the crystal is uniaxial, and if all are different it is biaxial. In all cases when calculating (21) we can put

$$\mathbf{D}.\mathbf{E} = K_1E_x^2 + K_2E_y^2 + K_3E_z^2 \quad . \quad . \quad (23)$$

§ 29. Minimum energy

Suppose that we are given certain conductors and each receives a given fixed total charge. We can prove that these charges will distribute themselves over the surfaces of the various conductors in such a way that the energy W is a minimum. For let $\mathbf{E}$ be the true field, and let $\mathbf{E}+\mathbf{E}'$ be the field that arises when the charges are held fixed in different positions on the conducting surfaces. In this condition the surfaces of the conductors will no longer be equipotentials, but since the total charge on each conductor is unaltered in the transformation from $\mathbf{E}$ to $\mathbf{E}+\mathbf{E}'$, we may apply Gauss' flux theorem to each conductor in the form

$$\int K\mathbf{E}.d\mathbf{S} = 4\pi \times \text{total charge on conductor} = \int K(\mathbf{E}+\mathbf{E}').d\mathbf{S}.$$

So

$$\int K\mathbf{E}'.d\mathbf{S} = 0 \quad . \quad . \quad . \quad . \quad (24)$$

We are supposing that there is no volume distribution of charge so that

$$\operatorname{div} K\mathbf{E} = 0 = \operatorname{div} K(\mathbf{E}+\mathbf{E}'),$$

i.e.

$$\operatorname{div} K\mathbf{E}' = 0 \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Our next step is to prove that the integral of $K\mathbf{E}.\mathbf{E}'$ through all space outside the conductors, is identically zero. In fact, if V is the potential due to $\mathbf{E}$,

$$\int K\mathbf{E}.\mathbf{E}'\,dv = -\int K\mathbf{E}'.\text{grad}\,V\,dv$$
$$= -\int \{\operatorname{div} VK\mathbf{E}' - V \operatorname{div} K\mathbf{E}'\}\,dv.$$

The last term on the right-hand side vanishes, by (25). The other term may be transformed by Green's theorem into an integral over the sphere at infinity together with integrals over the surface of each conductor. Considerations of order of magnitude show that the contribution from the sphere at infinity is zero for any finite system of charges; the contribution from any conductor is $\int KV\mathbf{E}'.d\mathbf{S}$, where $d\mathbf{S}$ is measured outwards from the conductor. Now V is constant

* The student of thermodynamics will recognise that this is the thermodynamic free energy and not ordinary energy, a distinction that is important if K depends on the temperature or the pressure.

on each such surface ; so, by (24) all these terms vanish also, and it follows that

$$\int K\mathbf{E}.\mathbf{E}'\,dv = 0.$$

But if $W+W'$ is the new energy

$$W' = \frac{1}{8\pi}\int (K(\mathbf{E}+\mathbf{E}')^2 - K\mathbf{E}^2)dv$$
$$= \frac{1}{8\pi}\int (2K\mathbf{E}.\mathbf{E}'+K\mathbf{E}'^2)dv$$
$$= \int \frac{K\mathbf{E}'^2}{8\pi}\,dv.$$

Since K is positive W' is always greater than zero showing that in the actual position taken up by the charges W is a minimum. Electrical energy therefore plays the same role here as potential energy does in mechanics.

§ 30. Stresses in the medium

According to Faraday the tubes of displacement $\mathbf{D}$ are in a state of stress ; it is this stress which gives rise to the energy $\frac{\mathbf{D}.\mathbf{E}}{8\pi}$ and is responsible for the apparent action-at-a-distance between charges. Let us see whether we can discover a stress system which will account for the inverse square law in an infinite medium of dielectric constant K. We shall find, as Maxwell discovered, that it is necessary to suppose that tubes of $\mathbf{D}$ are not only in a state of tension, but that they also exert a force perpendicular to their direction on neighbouring tubes.

Consider two charges $\pm e$ at A and B, a distance $2a$ apart (Fig. 24). According to (13) the force between them is $e^2/4Ka^2$ in the direction AB. If the tubes of $\mathbf{D}$ are in a state of stress, this is the force that they must exert across

any boundary which completely separates the charges. In particular this is the force across the mid-plane PO. We choose this plane because at all points P the tubes of force cross the plane at right angles and we have only to consider the tension in the tubes: the repulsion between neighbouring tubes will make no contribution to the force $e^2/4Ka^2$ for this is perpendicular to the plane PO.

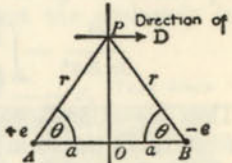


FIG. 24

Let $f(D)$ be the tension exerted across unit area of a tube where the displacement is $\mathbf{D}$. Our problem is to calculate the form of the function f.

Now the total pull across the mid-plane is $\int f(D)dS$, where dS is an element of area in the plane OP, and the integration extends over the whole plane. As D depends on OP and not on the azimuthal angle round AB, we can put

$$dS = 2\pi\, OP\, d(OP) = 2\pi a^2 \frac{\sin\theta}{\cos^3\theta}\,d\theta.$$

Thus

$$\frac{e^2}{4Ka^2} = \text{total pull across plane} = \int_0^{\pi/2} f(D)\, 2\pi a^2 \frac{\sin\theta}{\cos^3\theta}\,d\theta.$$

But the value of D at the point P is merely the sum of the values due to the charges $\pm e$ separately, so that $D = \frac{2e}{r^2}\cos\theta = \frac{2e}{a^2}\cos^3\theta$. Hence

$$\frac{e^2}{8\pi Ka^4} = \int_0^{\frac{\pi}{2}} f\left(\frac{2e\cos^3\theta}{a^2}\right)\frac{\sin\theta}{\cos^3\theta}\,d\theta.$$

This equation is to be true for all values of e. But e occurs in the form e^2 on the left-hand side; it must therefore occur in the same way on the right. This is only possible if

$f(D) = \lambda D^2$, where λ is some constant to be determined. The equation then gives

$$\frac{e^2}{8\pi K a^4} = \int_0^{\frac{\pi}{2}} \lambda \left(\frac{2e\cos^3\theta}{a^2}\right)^2 \frac{\sin\theta}{\cos^3\theta}\, d\theta = \frac{\lambda e^2}{a^4}.$$

Thus $\lambda = 1/8\pi K$, so that the tension $f(D)$ along a tube of force takes the form $D^2/8\pi K$, or $KE^2/8\pi$, or $(\mathbf{D}.\mathbf{E})/8\pi$.

We have still to discuss the force between neighbouring tubes. To do this we replace the charge $-e$ at B in Fig. 24 by a charge $+e$ and consider the Coulomb repulsive force between the two charges $+e$ at A and B. We leave it as an exercise to the student to show, by reasoning similar to that just used, that the appropriate force is obtained if each tube exerts on neighbouring tubes a repulsion $KE^2/8\pi$ per unit area.

These arguments have shown that in the particular case of two equal charges we need to assume a tension and pressure along tubes of force, each of amount $KE^2/8\pi$. As these forces only involve the local values of K and $\mathbf{E}$, it is reasonable to suppose that the same stress system would apply in other cases, and that the precise origin of the field is immaterial. It can indeed be proved that these stresses do give the correct mechanical forces whatever the charge distribution. But we shall content ourselves here with noticing that at the surface of a charged conductor where the tubes of force leave normally, we should expect them to exert on the conductor a mechanical force equal to $f(D)$, i.e. $KE^2/8\pi$ per unit area. This agrees exactly with the value previously deduced from Gauss' law in (12).

There is an interesting application of these results to the force at the boundary between a dielectric K and a vacuum. If the field is given, as in Fig. 25, by the continuous vectors D_n and E_s, it is seen that there is a mechanical force on the dielectric whose normal component is

$$\frac{1}{8\pi}\left\{D_n^2\left(1-\frac{1}{K}\right)+E_s^2(K-1)\right\},$$

directed from the dielectric into the vacuum. Since $K>1$ this force is always positive.

We have supposed in the above that the dielectric is incompressible. It is possible to show, though we shall not do so here, that if K varies with the pressure and density, there are other forces exerted in the dielectric by an electric field: one of their effects is to contract or expand the dielectric, the phenomenon known as **electrostriction**, by observation of which the theory may be tested.

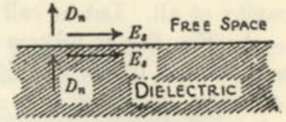


FIG. 25

§ 31. Cavities in a solid dielectric

We have hitherto defined $\mathbf{E}$ at any given point as the force on a unit charge placed at that point. This is quite satisfactory if the medium is a liquid or a gas for the charge can move and hence the force on it may be determined. But if the dielectric is solid, it is rather difficult to see what is meant by the definition of $\mathbf{E}$, since to measure it we have to excavate a small hollow in the medium and imagine our charge to be placed inside this hollow; it is then no longer in the dielectric medium! We are therefore led to consider the field inside such a cavity. There are two types of cavity that are particularly important; we may refer to them as needle-shaped and disc-shaped. In Fig. 26A we have supposed a cavity to be made having the shape of a thin disc whose plane is perpendicular to the direction of $\mathbf{E}$; in Fig. 26B the cavity is needle-shaped, pointing in the same direction as $\mathbf{E}$. Both cavities are small; all round them the dielectric constant is K, but inside them it is unity.

Now at points inside the dielectric, but fairly near the cavities, the electric field may be regarded as the sum of two parts. One part is the undisturbed field, which was there before the cavity was made, and the other is the

perturbing effect of the cavity. But if all the dimensions of the cavity are small, this latter contribution is small (this is verified in more detail later, §§ 75, 76), so that the field outside the cavity is effectively the same as if there were no cavity at all. Let us call this **E**.

In Fig. 26A it follows from symmetry that in the central part of the cavity the lines of force go straight across the

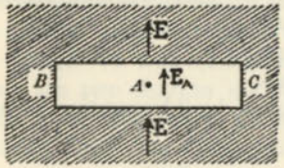


FIG. 26A.—Disc-shaped cavity; measures **D**.

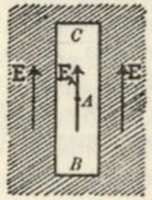


FIG. 26B.—Needle-shaped cavity; measures **E**.

gap; near the sides they will bend a little due to edge effects. But if the radius of the cavity is much greater than the thickness, the electric field near A may be regarded as uniform, and is related to **E** by means of the continuity conditions (18). Thus the field $\mathbf{E}_A$ in the central part of the cavity is directed across the cavity parallel to **E**, and from the fact that KE_n is continuous it follows that

$$\mathbf{E}_A = K\mathbf{E} = \mathbf{D},$$

so that the force on a unit charge placed in the cavity of the shape of Fig. 26A will measure the displacement **D** in the solid and not the field **E**. Similarly in Fig. 26B, the field at A will be parallel to **E** since the edge effects from B and C will be insignificant. So the boundary condition that the surface component of **E** is continuous tells us that

$$\mathbf{E}_A = \mathbf{E},$$

and the force on a unit charge in a needle-shaped cavity will indeed measure the internal field **E** in the solid. Cavities

of different shapes will measure different combinations of **D** and **E**. We shall show in Chapter IX, for example, (§ 76), that in a small spherical cavity the field is $3\mathbf{D}/(1+2K)$.

§ 32. Examples

1. Show from a consideration of the boundary conditions (18) that at a change of dielectric medium lines of force are refracted according to the law $K_1 \cot \theta_1 = K_2 \cot \theta_2$, where θ_1 and θ_2 are the angles between the directions of the field and the common normal to the boundary.

2. S_1 and S_2 are two equipotential surfaces completely surrounding an electrostatic system on which the net charge is Q; their potentials are V_1 and V_2 and there is no dielectric between them. Prove that if S_1 and S_2 are taken to be the plates of a condenser and the medium between them is of uniform dielectric constant K, the capacity is $KQ/(V_1 - V_2)$.

3. A spherical condenser consists of two concentric spheres of radii a and d. Concentric with these and lying between them is a spherical shell of dielectric K bounded by the spheres $r = b$, $r = c$. Show that if $a<b<c<d$, the capacity C of the condenser is given by

$$\frac{1}{C} = \frac{1}{a} - \frac{1}{d} + \frac{1-K}{K}\left(\frac{1}{b} - \frac{1}{c}\right).$$

4. Show that the polarisation vector between $r = b$ and $r = c$ in the previous question is $\dfrac{K-1}{4\pi K} \times \dfrac{Q}{r^2}$, directed radially, Q being the charge on either plate of the condenser. Show that the equivalent bound charges (see § 25) are $\rho = 0$ together with two surface layers

$$\sigma_b = -\frac{K-1}{4\pi K}\frac{Q}{b^2} \text{ and } \sigma_c = \frac{K-1}{4\pi K}\frac{Q}{c^2}$$

at the boundary of the dielectric.

5. A condenser is formed of the two spheres $r = a$, $r = b$ $(b > a)$ with uniform dielectric K. The dielectric strength of the medium (i.e. the greatest permitted field strength before it conducts) is E_0. Show that the greatest potential difference between the two plates, so that the field nowhere exceeds the critical value, is $E_0 a(b-a)/b$.

6. If in the previous question the potential difference is gradually increased beyond the critical value so that charge can flow into part of the dielectric, show that the condenser does not break down completely until the voltage is increased to $E_0(b-a)$. Investigate the distribution of charges when this latter condition is reached.

7. The slab of dielectric in Fig. 23 is partly removed from between the plates so that of the total area A an area x is covered with dielectric and $\mathrm{A}-x$ with free space. Assuming that the lines of force go straight from one plate to the other, show that the capacity C is given by

$$C = \frac{\mathrm{A}}{4\pi(a+b+c)} + \frac{bx\left(1-\frac{1}{K}\right)}{4\pi(a+b+c)\left(a+\frac{b}{K}+c\right)}.$$

Calculate the energy and show that if the charge on the plates is kept constant the dielectric is drawn into the condenser.

8. There is a point charge at the origin in an infinite dielectric medium K. Verify that the stress system of § 30 is such that there is no resultant force on the volume confined between the spherical surfaces $r=a$, $r=b$ and a diametral plane through the charge.

9. A conducting sphere of radius a in an infinite dielectric medium receives a charge Q. It is now divided in two by a diametral plane and the halves are slightly separated. Show by a consideration of the stresses between tubes of force that the one part exerts on the other a force $Q^2/8Ka^2$. Verify that this is exactly equivalent to the vector addition of forces $2\pi\sigma^2/K$ per unit area of surface, where σ is the surface density of charge.

10. A condenser consists of the conducting spheres $r=a$, $r=b$. The dielectric constant is independent of r but varies with direction. Write down the differential equation for the potential, and show that it is satisfied by the expression $V=A+B/r$, where A and B are constants. Show that B is related to the charge Q on the inner sphere $r=a$ by the equation $4\pi Q = B\int K d\omega$, $d\omega$ being an element of solid angle. Hence show that the capacity of the condenser is

$$\frac{ab}{4\pi(b-a)}\int K d\omega.$$

11. Show that in an electrostatic problem in which the potential depends solely upon the radial distance r, the differential equation for V is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 K\frac{dV}{dr}\right) = -4\pi\rho.$$

A charge e is placed at the origin in a medium in which the dielectric constant is $1+a/r$. Show that the potential is

$$V = \frac{e}{a}\log\frac{r+a}{r}.$$

12. A small particle of polarisability α is placed in an inhomogeneous field $\mathbf{E}$. Show that the force on it is $\frac{1}{2}\alpha$ grad E^2. (Note that this implies that a small dielectric particle moves towards places of higher field strength.)

13. Write out the steps required to obtain equation (12).

14. Show that the energy expended in polarising a dielectric is $\frac{1}{2}kE^2$ per unit volume. Deduce that the energy of the field, regarded as the sum of the energy $\frac{E^2}{8\pi}$ in the æther and $\frac{1}{2}kE^2$ in the dielectric, is $\int \frac{\mathbf{D}.\mathbf{E}}{8\pi}\,dv$.

15. V and V' are the potential functions due to two different distributions of charge on a given series of m conductors. There are no other charges present. Show that if $\mathbf{a}$ is the vector KV' grad $V-KV$ grad V', then div $\mathbf{a}=0$. Next, using the divergence theorem $\int \mathbf{a}.d\mathbf{S} = \int \text{div } \mathbf{a}\, dv$, deduce **Green's Reciprocal Theorem** (cf. § 23, ques. 3) that if the conductors carry charges Q_j, Q_j' and their potentials are V_j, V_j', then

$$\Sigma Q_j V_j' = \Sigma Q_j' V_j.$$

16. $S_1 \ldots S_m$ are m conductors carrying charges $Q_1 \ldots Q_m$ at potentials $V_1 \ldots V_m$. A new conductor S_{m+1} is introduced carrying a charge Q_{m+1}, and the new potentials are found to be $V_1', V_2' \ldots V_m'{}_{+1}$. The two fields are $\mathbf{E}$ and $\mathbf{E}'$. Show that $\int K\mathbf{E}.\mathbf{E}'dv = \int K\mathbf{E}'.\mathbf{E}'dv - 4\pi Q_{m+1}V_m'{}_{+1}$, the integration being over all space outside the $m+1$ conductors. Deduce that if the

new conductor is either uncharged or put to earth, the electrical energy is diminished (cf. § 17).

17. The volume charge distribution ρ in a given electrostatic system, and the total charge on each conductor, are kept unaltered, but the dielectric constant is changed from K to $K+K'$. K and K' are not necessarily constant throughout all space. The value of the field changes from $\mathbf{E}$ to $\mathbf{E}+\mathbf{E}'$. $\mathbf{E}'$ and K' are small quantities. Show that $\operatorname{div} K\mathbf{E}' = -\operatorname{div} K'\mathbf{E}$, and $\int K'\mathbf{E}.d\mathbf{S} = -\int K\mathbf{E}'.d\mathbf{S}$ over each conductor. Further, writing $\mathbf{E} = -\operatorname{grad} V$, and using Green's theorem, prove that $\int K\mathbf{E}.\mathbf{E}'dv = -\int K'\mathbf{E}.\mathbf{E}\,dv$ and deduce that the change in energy is $-\int \frac{K'E^2}{8\pi}\,dv$. (Notice that this proves **Thomson's theorem** that an increase in the dielectric constant without alteration of charges decreases the energy of the field.)

18. Show that the field inside the cavities of Fig. 26 in § 31 may be obtained by the use of the apparent charges discussed in § 25.

CHAPTER V

STEADY CURRENTS

§ 33. The current vector

In our previous chapters we have considered only problems in which the electric charges were at rest: if charges are moving we say that there is a **conduction current.** This current is measured by the rate at which charge crosses unit area. In metals (see § 2) the conduction current is carried entirely by the conduction electrons; but in electrolytes both positive and negative charges are moving and the total conduction current is the algebraic sum of the two individual currents. In this chapter we shall only be concerned with **steady currents,** that is, problems in which the current flowing is independent of the time, though it may vary from place to place. Suppose that at a point P the charges are moving with mean velocity $\mathbf{v}$; then if there are N charged particles per unit volume and they each carry a charge e, we define the **conduction current vector,** or **current density** $\mathbf{j}$, at P by the relation

$$\mathbf{j} = Ne\mathbf{v} \quad . \quad . \quad . \quad . \quad (1)$$

The direction of $\mathbf{j}$ is the direction in which the current flows; the magnitude of $\mathbf{j}$ is a proper measure of current density, for if $d\mathbf{S}$ is an element of area (Fig. 27) all the charge within the tiny cylinder standing on $d\mathbf{S}$ as base and having generators of length v will cross $d\mathbf{S}$ in unit time. The volume of this cylinder is $\mathbf{v}.d\mathbf{S}$, so that the total charge involved is $Ne\mathbf{v}.d\mathbf{S}$, i.e. $\mathbf{j}.d\mathbf{S}$. Following the precedent of § 9, we call this the flux of $\mathbf{j}$ across $d\mathbf{S}$.

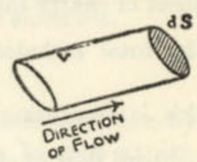


FIG. 27

We can similarly speak of the flux of $\mathbf{j}$ across any larger surface; this quantity will be given by the surface integral

$$\int \mathbf{j}.d\mathbf{S}. \quad . \quad . \quad . \quad . \quad (2)$$

It is important to realise that our definition of $\mathbf{j}$ shows it to be a vector quantity. We must not therefore confuse it with the total current i flowing in a given wire, a physical phenomenon with which we are all quite familiar. In fact, this latter quantity may be obtained by calculating the flux of $\mathbf{j}$ across any surface cutting through the wire. Thus

$$i = \int \mathbf{j}.d\mathbf{S}, \quad . \quad . \quad . \quad . \quad (3)$$

where the integration is over any cross-section. It may or may not happen that $\mathbf{j}$ is constant over this cross-section.

Since $\mathbf{j}$ is a vector quantity we shall have lines and tubes of $\mathbf{j}$ just as we have previously discussed lines and tubes of $\mathbf{E}$ and $\mathbf{D}$. We describe these as **lines of flow**, and **tubes of flow**. The differential equation of the lines of flow is

$$\frac{dx}{j_x} = \frac{dy}{j_y} = \frac{dz}{j_z} \quad . \quad . \quad . \quad . \quad (4)$$

Let us draw any closed surface S and consider the balance-sheet of charge inside S. If ρ is the volume density of charge the total included charge is $\int \rho dv$, and the rate at which this is decreasing is $-\int \frac{\partial \rho}{\partial t} dv$. This decrease is due to the outward flow of charge from S, so that by (2)

$$-\int \frac{\partial \rho}{\partial t} dv = \int \mathbf{j}.d\mathbf{S} \quad . \quad . \quad . \quad . \quad (5)$$

Transforming the right-hand side by the divergence theorem we get

$$\int \left(\operatorname{div} \mathbf{j} + \frac{\partial \rho}{\partial t}\right) dv = 0.$$

Since this result holds true for any volume, the integrand must vanish identically. This proves the **equation of continuity**

$$\operatorname{div} \mathbf{j} = -\frac{\partial \rho}{\partial t}. \quad . \quad . \quad . \quad . \quad (6)$$

An exactly similar equation occurs in the flow of liquids.*

The present chapter is solely concerned with steady conditions, so that $\frac{\partial \rho}{\partial t}$ vanishes. In this case the equation of continuity simplifies to

$$\operatorname{div} \mathbf{j} = 0 \quad . \quad . \quad . \quad . \quad (7)$$

The flux equation (5) needs modification if our surface S encloses any source of current. Places where current enters or leaves a conductor are known as positive and negative **electrodes**. If we draw S to surround an electrode the total flux out of S must equal the current i flowing from the electrode, sometimes called the **strength** of the electrode. Thus

$$\int \mathbf{j}.d\mathbf{S} = i. \quad . \quad . \quad . \quad . \quad (8)$$

This current i is clearly equal to the total strength of all the tubes of flow that start on the surface of the electrode.

§ 34. Conductivity

We must now consider in more detail the means whereby a current flows. In the first place, as can be seen experimentally by joining together two charged conductors, currents flow from regions of higher potential to regions of lower

* Cf. Rutherford, *Vector Methods*, 1946, p. 104.

potential. As we are here concerned with steady flow, the potential will have a definite constant value at any given point, the necessary differences of potential at the electrodes being maintained by an accumulator or battery.

In the second place the velocity $\mathbf{v}$ of the charges, introduced in (1), is surprisingly small. For example,* in a wire of 1 sq. mm. section carrying a current of 1 ampere, the value of $\mathbf{v}$ is of the order of 1 cm. per second. This is very much less than the random velocities of the electrons which are of the order of 10^7 cms. per second. This is important for two reasons: one is that the flowing of even a large current makes very little difference to the general distribution of velocities, so that many quantities, e.g. the time between two collisions, are unaffected by the current; the other is that the mean velocity $\mathbf{v}$ of the charges lies in the same direction as the field $\mathbf{E}$ causing the motion. This would not be the case if, for example, the electrons were able to acquire a large drift momentum, for when the direction of $\mathbf{E}$ changed, this momentum would resist any change in the direction of flow. As it is, we have the result that $\mathbf{j}$ and $\mathbf{E}$ are parallel vectors.

Experimentally it is found that $\mathbf{j}$ is proportional to $\mathbf{E}$, and we write

$$\mathbf{j} = \sigma\mathbf{E}, \quad . \quad . \quad . \quad . \quad (9)$$

where σ is the **conductivity** of the metal. The conductivity varies with the temperature, but is independent of the shape of the conductor. We can understand this law, which is known as **Ohm's law**, by the following argument, due to Drude.

Consider a particular conduction electron. There is an average time τ between successive collisions of this electron with the non-conducting material of the metal. If we suppose that after each such collision the electron rebounds in a perfectly random direction, then the drift velocity $\mathbf{v}$ in the direction of the field represents the mean additional velocity acquired

* See Jeans, *Electricity and Magnetism*, 1927, p. 307.

by the electron in the interval τ between successive collisions. If the field is $\mathbf{E}$ and the electron has charge e and mass m the force on it is $e\mathbf{E}$, so that its acceleration is $e\mathbf{E}/m$ in the direction of flow. This acceleration does not vary with the time, so that the mean drift velocity $\mathbf{v}$ is given by $\mathbf{v} = \frac{1}{2}\frac{e\mathbf{E}}{m}\tau$. But, from (1), $\mathbf{j} = Ne\mathbf{v}$; so eliminating $\mathbf{v}$,

$$\mathbf{j} = \frac{Ne^2\tau}{2m}\mathbf{E}.$$

This is Ohm's law, with the condition $\sigma = Ne^2\tau/2m$. For copper σ is about 5×10^{17} in e.s.u.,* and $e^2/2m$ is about $1{\cdot}3\times10^8$, so we should require $N\tau = 4\times10^9$ approximately. The number N of free electrons is about 2×10^{22} per c.cm., so that τ is roughly 2×10^{-13} secs. This is the time between successive collisions. If we assume, on other grounds, that the average velocity is about 10^7 cms. per second, this gives a distance of about 2×10^{-6} cms. between collisions. Since the distance apart of neighbouring nuclei is about 2×10^{-8} cms., this explanation of Ohm's law is entirely reasonable. It is true that recent quantum theory has thrown doubt on several details of the arguments above, but it is still true to say that this represents in essence a correct picture of the conduction current in a metallic conductor. In particular it explains why $\mathbf{j}$ is proportional to $\mathbf{E}$, and why the conductivity is a constant dependent only on the material of the conductor. The quantity $1/\sigma$ is known as the **specific resistance**, the reason for which we shall see shortly.

In non-isotropic conductors, e.g. certain crystals, the current may flow more easily in some directions than in others. In such cases we may choose our directions of x, y, and z so that (9) is replaced by the three equations

$$j_x = \sigma_1 E_x,\ j_y = \sigma_2 E_y,\ j_z = \sigma_3 E_z.$$

We shall not, however, discuss such problems in this book.

* This as in column 3 of table 1 on p. 242, is in units of $\sec^{-1}$.

§ 35. Differential equations of the field and flow

We are now in a position to summarise the equations that determine the field and flow. Thus:

$$\text{(i)}\quad \mathbf{E} = -\operatorname{grad} V. \qquad (10)$$
$$\text{(ii)}\quad \mathbf{j} = \sigma\mathbf{E}, \qquad (9)$$
$$\text{(iii)}\quad \operatorname{div} \mathbf{j} = 0. \qquad (7)$$

Combining (i), (ii) and (iii) we have the differential equation for V:

$$\text{(iv)}\quad \operatorname{div}(\sigma \operatorname{grad} V) = 0 \qquad (11)$$

In the particular case when $\sigma =$ constant, this last equation reduces to Laplace's equation

$$\nabla^2 V = 0 \qquad (12)$$

In addition we have the boundary conditions:—

(v) V is continuous, and at the surface of an electrode where a battery is providing charge at a definite potential,

$$V \text{ is constant} \qquad (13)$$

(vi) This last condition, combined with (i) and (ii), shows that current leaves an electrode normally, and we have already seen that if the total current leaving an electrode is i, then

$$\int \mathbf{j}.d\mathbf{S} = i \qquad (8)$$

(vii) At the boundary between a conductor and an insulator or vacuum there can be no normal flow of current, so that

$$j_n = 0. \qquad (14)$$

An example of this last result is found in a wire, where, however bent the wire may be, the normal component $j_n = 0$ at all parts of the wire's surface.

Equations (i) to (vii) constitute the differential equations of steady flow.

§ 36. Resistance

Suppose that we have a conductor of any shape with a current i entering at one electrode and leaving at another. If the difference of potential between the two electrodes is V, we define the **resistance** R of the conductor as the ratio

$$R = V/i \qquad (15)$$

It is easy to see that R is a property of the conductor and not of the current flowing through it, since if $\mathbf{j}$ is everywhere increased by a constant factor k, both V and i will be similarly increased so that V/i remains constant. Equation (15) is sometimes referred to as Ohm's law. But it is less fundamental than (9), from which indeed it follows.

Consider for a moment the particular case of a straight wire of length l and constant cross-section A, the electrodes being plane sections perpendicular to the wire, with potentials V_1 and V_2. All the conditions (i) to (vii) are satisfied if $\mathbf{j}$ is constant throughout the wire and equal in magnitude to $\sigma(V_1-V_2)/l$. The total current is $i = jA = \sigma A(V_1-V_2)/l$. Hence according to (15) the resistance must be $l/\sigma A$.

If the wire is a block of unit sectional area and unit length, the resistance is simply $1/\sigma$; this justifies our calling $1/\sigma$ the specific resistance in § 34.

§ 37. Heat loss

It is easy to see that there is a loss of electrical energy when a current flows: for charge is continually falling from regions of higher potential to regions of lower. We can calculate this loss of energy, which shows itself in the generation of heat, by considering (see Fig. 28) what happens in a small element AB of a tube of flow whose length is $d\mathbf{s}$ and cross-section $d\mathbf{S}$. In unit time a total charge jdS flows into the volume at A and an equal charge flows out at B. There is a drop of potential $V_A - V_B$, and

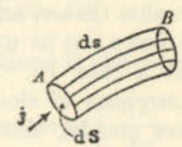


Fig. 28

hence the rate of loss of electrical energy in this small volume is $(V_A - V_B)j\ dS$. But $V_A - V_B = E\ ds$, and the volume element AB is $dsdS$, so that the rate of loss of electrical energy * is jE per unit volume. Since $\mathbf{j}$ and $\mathbf{E}$ are in the same direction we may write this $\mathbf{j}.\mathbf{E}$. The total rate of loss is therefore

$$\int \mathbf{j}.\mathbf{E}\, dv = \int \sigma E^2 dv \quad . \quad . \quad . \ (16)$$

We can transform (16) by using the divergence theorem. Thus

$$\int \mathbf{j}.\mathbf{E}\, dv = -\int \mathbf{j}.\text{grad } V\, dv$$
$$= -\int \{\text{div } V\mathbf{j} - V \text{ div } \mathbf{j}\} dv.$$

The integral of the second term on the right-hand side vanishes since, by (iii), div $\mathbf{j} = 0$. The integral of the first term may be transformed by Green's theorem, and gives $-\int V\mathbf{j}.d\mathbf{S}$, where the integration covers the entire surface of the conductor. At a boundary between the conductor and a vacuum or a non-conductor, by (vii) $j_n = 0$; at each electrode, from (v) and (vi), remembering that the direction of $d\mathbf{S}$ is out of the conductor, i.e. into the electrode, we get a contribution Vi, where V is the potential of the electrode and i the current flowing out from it. So the total loss of energy per unit time is ΣVi.

A simple explanation of this formula is obtained if we realise that at an electrode of potential V from which a charge i flows out in unit time, electrical energy is being provided at a rate Vi. ΣVi therefore represents the total rate of provision of electrical energy, and hence, since the currents are steady, this must also be the rate at which electrical

* Measuring $\mathbf{j}$ and $\mathbf{E}$ in electrostatic units. A constant of proportionality is required if, as usually happens in practice, $\mathbf{j}$ is measured in electromagnetic units. See Chapter VI.

energy is being converted into heat energy. It is sometimes referred to as the **Joule heat loss.** The mechanism of this change of energy is soon recognised from our description in § 34. During the time between two collisions, each conduction electron acquires a forward momentum in the direction of the current. At the moment of collision this is communicated to the fixed substance of the conductor, the nuclei of which will therefore vibrate with more energy. According to the Kinetic Theory this energy of vibration may be described as heat, and reveals itself in a rise of temperature.

If our conductor has just two electrodes at potentials V_1 and V_2, the currents flowing out from them must be $+i$ and $-i$, so that the Joule heat loss may be written in either of the forms

$$(V_1 - V_2)\, i = i^2 R = (V_1 - V_2)^2/R \quad . \quad . \ (17)$$

Students of physics will recognise this formula as the one generally used in calculating the heat produced in a wire carrying a current.

§ 38. Comparison with electrostatics

If we compare the equations of current flow (i) to (vii) of § 35, with the equations of the electrostatic field in Chapter IV, we shall notice that the two sets are exactly equivalent mathematically, provided that we put $\rho = 0$ in the electrostatic problem. Thus, putting corresponding quantities in parallel columns:

$\mathbf{j} = \sigma\mathbf{E}$	$\mathbf{D} = K\mathbf{E}$,
div (σ grad V) $= 0$	div (K grad V) $= 0$,
electrode equation $\int \mathbf{j}.d\mathbf{S} = i$	Gauss' equation $\int \mathbf{D}.d\mathbf{S} = 4\pi Q$,
heat loss $\int \sigma E^2 dv$	electrostatic energy $\int \frac{KE^2}{8\pi}\, dv$.

It therefore follows that any theorem proved for electrostatics has an exact analogue in the flow of steady currents. In

particular, if we have solved the problem of a condenser with two plates at definite potentials V_1 and V_2, carrying charges $\pm Q$, so that the capacity C is $Q/(V_1-V_2)$, we get immediately the solution for a steady flow of current between two electrodes at potentials V_1 and V_2 in a medium whose conductivity has the same numerical value as the previous dielectric constant. The equations listed above show that the current i is numerically equal to $4\pi Q$. But the resistance R is $(V_1-V_2)/i$ which clearly has the same numerical value as $(V_1-V_2)/4\pi Q$, that is $1/4\pi C$. Thus the resistance of a conductor corresponds in the mathematical analysis to $1/4\pi C$.

The equivalence of the mathematics for the two types of problem enables us to state at once :

(*a*) If given total currents flow from certain electrodes the current density in the medium distributes itself in such a way that the Joule heat loss is a minimum (cf. § 29).

(*b*) If the conductivity of any part of a conductor is increased, the resistance of the whole conductor will be decreased (cf. § 32, ques. 17).

(*c*) If, when a series of electrodes are kept at potentials V_1, V_2, . . . the currents leaving them are i_1, i_2, . . . : and if when the potentials are changed to V_1', V_2', . . . the currents are i_1', i_2', . . . , then $\Sigma iV' = \Sigma i'V$ (cf. § 32, ques. 15. This is another case of Green's reciprocal theorem.)

The student is advised to prove these results directly, without reference to the earlier work ; this will familiarise him with the technique and fundamental equations of this chapter.

§ 39. A worked example

In media where the conductivity is constant, the potential equation (11) is merely Laplace's equation $\nabla^2 V = 0$. If we take any solution of this equation, $V = V(x, y, z)$, it gives us the solution of a problem in current flow between electrodes whose shape coincides with any two of the equipotential surfaces $V(x, y, z) =$ constant.

In plane polar coordinates (r, θ) the function $V = c\theta$ is a solution of $\nabla^2 V = 0$, if c is a constant. Equipotential surfaces are given by $\theta =$ constant. The field is given by $\mathbf{E} = -\text{grad } V$. Its magnitude is c/r and it is directed perpendicular to the radius vector. Thus the current $\mathbf{j} = \sigma\mathbf{E}$ flows in circles about the origin. This is clearly the solution for the flow of current in a thin strip (shown shaded in Fig. 29), lying in the xy plane, bounded by any two concentric circles and having the lines AB, CD as electrodes.

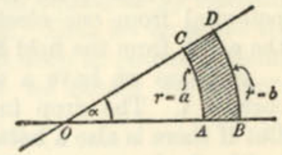


FIG 29.

The total current leaving AB $(\theta = 0)$ is $\int \mathbf{j} . d\mathbf{S}$. If t is the thickness of the strip, then since $j = -\frac{c\sigma}{r}$, this becomes

$$-c\sigma t \int_a^b \frac{dr}{r} = -c\sigma t \log \frac{b}{a}.$$

But the difference of potential between the electrodes AB and CD is $-c\alpha$, where $\alpha = \angle AOC$. Hence the resistance of the strip is

$$\alpha \Big/ \left\{\sigma t \log \frac{b}{a}\right\}.$$

§ 40. Networks

We have not so far referred in detail to the mechanism whereby a potential difference is provided between two electrodes. The most convenient source of potential is a **battery, or accumulator,** in which the energy needed to make the current flow arises from chemical action between the plates, or terminals, of the battery. For our purposes it is sufficient to regard a battery as a discontinuity in potential, and we shall speak of the **electromotive force** (e.m.f.) of a battery as the difference in potential between its

terminals. If the battery is joined to the two electrodes of a conductor, and if we may neglect any internal resistance in the battery, the e.m.f. must be exactly equal to the drop in potential from one electrode to the other. To distinguish the e.m.f. from the field $\mathbf{E}$, we shall label it $\mathcal{E}$.

Suppose we have a wire AB of resistance R carrying a current i. The drop in potential between its ends is Ri. But if there is also a battery $\mathcal{E}$ (Fig. 30), the drop of potential is $Ri-\mathcal{E}$, since on crossing the battery in the direction from A to B there is an increase of $\mathcal{E}$ in the potential.

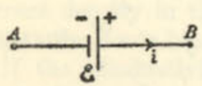


FIG. 30

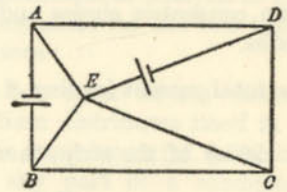


FIG. 31

If we have a series of wires joined together in any way, we describe the result as a **network**. Some or all of the wires may contain batteries. If current is generated wholly or partly by outside batteries we call it an **open network**; but if the system is complete in itself with no outside connections it is a **closed network**. Fig. 31 shows an example of a closed network containing two batteries, in the arms AB and DE.

Let the current in AB be called i_{AB} and be positive if the direction is from A to B. Then $i_{AB} = -i_{BA}$. Similarly let the total e.m.f. in AB be $\mathcal{E}_{AB}$, and be positive if, as in Fig. 30 the positive terminal is towards B. Thus, in Fig. 31, $\mathcal{E}_{AB}$ and $\mathcal{E}_{DE}$ are positive, all the others being zero.

Since no current is being stored up at any point of the network, the total current leaving any juncture such as A must be zero. Thus

$$i_{AB}+i_{AE}+i_{AD} = 0 \quad . \quad . \quad . \quad (18)$$

This is **Kirchhoff's first law**; there is a similar equation for each of the points in the network.

We can establish another series of equations by selecting any complete circuit in the network. Thus let us select the circuit $ABEA$. Evidently the total drop in potential round this circuit must be zero. Hence we obtain **Kirchhoff's second law**:

$$\Sigma(Ri-\mathcal{E}) = 0, \quad . \quad . \quad . \quad (19)$$

the summation being over all the elements in the circuit taken in a continuous direction round the circuit. Each possible circuit gives a separate equation. However not all these are independent. For example, the circuits $ABEA$ and $BCEB$ give two equations which when added together are equivalent to the equation from $ABCEA$. The complete system of equations provided by Kirchhoff's two laws is sufficient to solve completely the problem of current distribution in the arms of the network. The theory of networks has been developed quite extensively by electrical engineers, but we shall be content with one example to illustrate the way in which calculations are made.

§ 41. The Wheatstone Bridge

Consider the closed network in Fig. 32, known as Wheatstone's Bridge. The only battery is in the arm AC, supposed

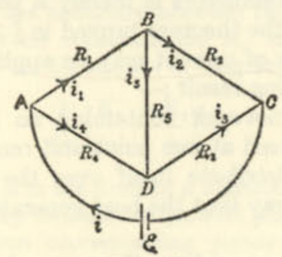


FIG. 32

to have zero resistance. Let the other resistances be R_1, R_2, . . . R_5 as shown, and the corresponding currents

$i_1, i_2 \ldots i_5$; i being the total current taken from the battery $\mathcal{E}$.

Applying Kirchhoff's first law at A, B, C and D in turn, we get

$$i_1+i_4 = i, \quad i_1 = i_2+i_5,$$
$$i_2+i_3 = i, \quad i_4 = i_3-i_5.$$

Applying the second law to the circuits $ABDA$, $BCDB$ and $ABC\mathcal{E}A$, we get

$$R_1i_1+R_5i_5 = R_4i_4, \quad R_2i_2-R_5i_5 = R_3i_3, \quad R_1i_1+R_2i_2 = \mathcal{E}.$$

When these equations are solved the currents are known in all branches of the network. The most interesting case is that in which the resistances are adjusted so that no current flows through BD. This is generally tested by having an ammeter or galvanometer in this arm. Then

$$i_1 = i_2, \quad i_3 = i_4, \quad i_5 = 0,$$
$$R_1i_1 = R_4i_4, \quad R_2i_2 = R_3i_3.$$

Eliminating the ratios of the currents we obtain the familiar condition which gives us any one resistance in terms of the other three:

$$R_1R_3 = R_2R_4 \quad . \quad . \quad . \quad . \quad (20)$$

§ 42. Heat generated in a network

A network of conductors is merely a particular case of a conductor, so that the theorem proved in § 38 (*a*) for the heat generated by a flow of current may be applied directly. This gives us the following result :—

If in an open network containing no batteries, a total current i is introduced at one point and removed at another, the current will distribute itself over the branches of the network in such a way that the heat generated in the network is a minimum.

There is a corresponding theorem which applies in a closed network. The result, which we shall not prove, is that

$$\Sigma i_{AB}(R_{AB}i_{AB}-2\mathcal{E}_{AB}) \text{ is a minimum} \quad . \quad . \quad (21)$$

§ 43. Examples

1. Twelve equal wires of resistance R are joined at their ends to form the edges of a cube. If current enters and leaves at opposite corners of the cube, show that the resistance is $5R/6$, and if it enters and leaves at two ends of one wire the resistance is $7R/12$.

2. A set of resistances $R_1 \ldots R_n$ are joined in series (c.f. § 21). Show that the total resistance R equals $R_1 + \ldots + R_n$. But if they are joined in parallel

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}.$$

3. Show, on the basis of Drude's theory of conduction (§ 34) that the rate at which energy is given up by the conduction electrons to the non-conducting material of the conductor is $\mathbf{j}.\mathbf{E}$ per unit volume. Notice that this provides another proof of the Joule heat loss (16).

4. Show that in steady flow the strength of a tube of flow is constant along its length.

5. Show that lines of flow are refracted at the boundary between two media of different conductivities.

6. Show that if a steady current with normal component j_n is flowing across the boundary between two conducting media in which the dielectric constants are K_1 and K_2, there must be a charge density on the surface of magnitude

$$\frac{1}{4\pi}\left(\frac{K_2}{\sigma_2} - \frac{K_1}{\sigma_1}\right)j_n.$$

7. A conductor is in the form of a cylinder of arbitrary cross-section S bounded by two almost parallel plane sections. If t is the distance between corresponding points on the two ends, show that the resistance R is given by

$$\frac{1}{R} = \sigma \int \frac{dS}{t}.$$

8. Prove equation (21) that if in a closed network of wires there are batteries of e.m.f. $\mathcal{E}_{AB}$ then the currents distribute themselves in such a way that $\Sigma i_{AB}\,(R_{AB}i_{AB}-2\mathcal{E}_{AB})$ is a minimum.

9. Use the result of equation (21) applied to a closed network of wires in which there are batteries of fixed e.m.f. to show that if any two points of a network are joined by a wire of arbitrary resistance the total heat generated is increased. Show also that if the original network is an open one in which there are no batteries its resistance is decreased by joining any two points.

10. Prove the theorems (*a*), (*b*), (*c*) of § 38 for the flow of current without quoting the corresponding theorems proved in electrostatics.

11. Use Green's reciprocal theorem (§ 23, ques. 3) to show that if AB and CD are two arms of a closed network containing no batteries, then the current in AB when an e.m.f. $\mathcal{E}$ is introduced in CD is the same as the current in CD when the e.m.f. is introduced in AB. If a battery in AB produces no current in CD, AB and CD are called **conjugate conductors**. Show that in this case a current entering the network at A and leaving at B produces no current in CD.

12. C is a closed curve in the xy plane lying entirely on the positive side of the y axis. This curve is now rotated through 180° about the y axis. The volume between the curved surface so formed and the two plane ends is filled with a uniform conducting material. If when the two ends are electrodes the resistance is R, show that $\frac{1}{R}=\frac{\sigma}{\pi}\int\frac{dS}{x}$, where the integration is over the area embraced by the curve C.

13. Show that the potential function $V=A\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$ represents the flow of current between two small spherical electrodes in an infinite conducting medium, r_1 and r_2 being distances from the centres of the two electrodes. If the electrodes are of small radii c_1 and c_2 placed a distance a apart show that the resistance between them is approximately R_1+R_2, where $R_1=(a-c_1)/4\pi\sigma ac_1$, $R_2=(a-c_2)/4\pi\sigma ac_2$. This result allows us to speak of an **electrode resistance** R associated with one electrode and independent of the other. R tends to infinity as c tends to zero.

14. A and B are opposite ends of a diameter AOB of a thin spherical shell of radius a and thickness t. Current enters and leaves by two small circular electrodes of radius c whose centres are at A and B. If i is the total current and P is a point on the shell such that the angle $POA=\theta$, show that the magnitude of the current vector at P is $i/(2\pi at\sin\theta)$. Deduce that the resistance of the conductor is $\frac{1}{\pi\sigma t}\log\cot\frac{c}{2a}$.

15. The space between two coaxial cylinders of radii a and b and of length t is filled with a medium of conductivity σ. Show that the resistance between the two cylinders is $\frac{1}{2\pi\sigma t}\log\frac{b}{a}$.

16. What problems in three-dimensional steady flow of current are solved by the potential functions (i) $V=Axy$, (ii) $V=\frac{Ax}{(x^2+y^2+z^2)^{\frac{3}{2}}}$?

17. Show that if the arcs AC and BD in Fig. 29 are taken to be two electrodes the resistance of the conductor is changed to $\frac{1}{\sigma at}\log\frac{b}{a}$.

18. A and B are opposite ends of a diameter of a thin circular disc of radius a and thickness t. Current enters and leaves the disc by two circular wires whose centres are at A and B and whose radius c is much less than a. Show that the distribution of current may be obtained by taking $V=b\log r_1/r_2$, where r_1 and r_2 are distances from A and B. Deduce that the lines of flow are circular arcs through A and B, and that the resistance of the disc is $\frac{2}{\pi\sigma t}\log\frac{2a}{c}$.

19. AB is a uniform telegraph wire. At some unknown point C of the wire there is a fault, i.e. a resistance of unknown

magnitude connecting C to the earth. B is put to earth potential and an e.m.f. is applied at A. Next A is put to earth potential and the e.m.f. is applied at B. The total resistance is measured in each case. Show how to determine the position of C from these measurements.

20. In a uniform submarine cable there is a leak resistance r per unit length and the resistance of the cable is R per unit length. At distance x along the cable the potential is V and the current is i. Show that $\frac{dV}{dx} = -Ri$ and $\frac{di}{dx} = -\frac{V}{r}$. Deduce a differential equation for V, and show that if the cable is of length l the two ends being at potentials V_0 and 0, then

$$V = V_0 \operatorname{cosech} \sqrt{\frac{R}{r}}\, l \sinh \left(\sqrt{\frac{R}{r}}\,(l-x)\right).$$

CHAPTER VI

MAGNETIC EFFECTS OF CURRENTS

§ 44. Magnetic effects of a small coil

Our discussion in Chapter V was solely concerned with the flow of current within one single conducting system; if we consider two distinct systems we discover that these exert forces on one another. The discovery of this effect goes back to Oersted (1820) who found that an electric current could exert forces precisely similar to those exerted by the so-called permanent magnets. We therefore speak of these effects as the magnetic effects of a current. In this chapter we shall deal with such effects when the medium in which they take place is a vacuum. However, just as in the electrostatics of Chapters II and III, very little difference occurs if the medium is ordinary air. The actual difference is discussed in Chapter VII.

The full investigation of these mutual effects between two currents is due chiefly to Ampère. His conclusions may be summarised as follows :—

(i) Two small coils carrying currents exert forces on each other of precisely the same type as those exerted by two electrical dipoles. This means that each coil experiences both a force and a torque, and these have the same dependence on angle and distance as is involved in eq. (18) of Chapter III.

(ii) If the wires bringing the current to and from the coil are very close together, they contribute nothing to these forces between the coils. The forces therefore are solely due to what happens in and near the coils themselves.

(iii) Eq. (18) in Chapter III shows that the absolute magnitude of the forces depends on two vector quantities $\mathbf{m}_1$ and $\mathbf{m}_2$. We must therefore associate with each small loop a vector, known as the **magnetic moment** of the loop. The correct angular variation of force and torque is obtained if the magnetic moment is directed perpendicularly to the plane of the loop, and points along the positive direction of the normal. This direction is such that a right-handed rotation about it takes us round the loop in the same direction as the current in the loop. An individual measurement of force or couple between two coils gives us the product $m_1 m_2$. To get m_1 or m_2 alone we must consider three small coils; taking them in pairs we determine the products $m_1 m_2$, $m_2 m_3$, $m_1 m_3$, from which m_1, m_2 and m_3 are each separately found.

(iv) The magnetic moment $\mathbf{m}$ of a small loop carrying a current i is proportional to the current i and also to the area of the loop dS. It is independent of the shape, provided that the area is constant. We may therefore write $\mathbf{m} = k\, i\, d\mathbf{S}$, where k is some universal constant.

§ 45. Electromagnetic units

This last result gives us a new way of measuring current. We define the c.g.s. electromagnetic unit of current by putting $k = 1$, so that unit current gives a magnetic moment $\mathbf{m}$ equal to $d\mathbf{S}$. In general, with current i,

$$\mathbf{m} = i\, d\mathbf{S} \quad . \quad . \quad . \quad . \quad (1)$$

Units based on this definition are called **electromagnetic units** (e.m.u.), since they depend on magnetic effects. The student may wonder why it is necessary to have two different sets of units. The explanation is partly that before 1820 the subjects of electrostatics and magnetism grew up quite separately, and only later were seen to be related; partly also that, because currents are usually measured by their magnetic effects and charges by their electrostatic effects, it is convenient to have units simply related to the type of

measurement usually made. The two systems are of course related. If we use the c.g.s. system it turns out that

$$1 \textit{ e.m.u. of current is approximately } 3 \times 10^{10} \textit{ e.s.u. of current} \quad (2)$$

We shall have more to say about this ratio, which is known as c, in Chapters XIII and XIV, where we shall see that it represents a fundamental velocity. For the rest of this chapter, however, we shall use e.m.u., and adopt (1) as our definition of current i.

§ 46. Magnetostatic field

It is a natural development from our earlier work to introduce a field whereby the mutual interaction between two circuits may be communicated. Indeed, if like Faraday we refuse to accept the idea of action-at-a-distance, we are led automatically to a **magnetic field**, represented by some vector $\mathbf{H}$. We cannot, of course, define $\mathbf{H}$ just as we defined $\mathbf{E}$, since there is no magnetic particle corresponding to an isolated electrostatic charge. But we can say that there is a magnetic field $\mathbf{H}$ at any point P if a little coil, sometimes called a **search coil**, carrying a current experiences a torque when placed at P. And we can measure $\mathbf{H}$ in terms of this torque; in fact, using eq. (16) of Chapter III, $\mathbf{H}$ is determined by the fact that the couple exerted by the field on a small coil of magnetic moment $\mathbf{m}$ is given, both in magnitude and direction by

$$\text{couple} = \mathbf{m} \times \mathbf{H} \quad . \quad . \quad . \quad . \quad (3)$$

$\mathbf{H}$ is usually called the magnetic field, but in steady cases such as we are discussing in this chapter a better title is **magnetostatic field.** This field is a vector field, so that all the discussion of lines of force, tubes of force and unit tubes, which we have given in relation to $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{j}$, will apply equally well to the magnetostatic field.

Thus, in equation (17) of Chapter III we showed that the potential energy of an electric dipole $\mathbf{m}$ in a field $\mathbf{E}$ was $-\mathbf{m}.\mathbf{E}$. It follows that in the presence of a magnetic field $\mathbf{H}$, the

potential energy of a small electric coil with constant current i is $-\mathbf{m}.\mathbf{H}$. Thus,

$$\text{potential energy} = -i\,\mathbf{H}.\mathbf{dS} \qquad . \quad . \quad . \quad (4)$$

It also follows, by analogy with electrostatics (see also § 50), that **H** is derivable from a potential Ω, and we may write

$$\mathbf{H} = -\text{grad}\,\Omega, \quad . \quad . \quad . \quad . \quad (5)$$

where Ω is called the **magnetostatic potential.**

§ 47. Magnetic poles

The discussion of the magnetic field due to currents has so far followed much the same lines as our earlier discussion of the electrostatic field, except that we have always used parallels involving electric dipoles. The question immediately arises, whether there is any magnetic particle that corresponds to a point electric charge. This is particularly important because we defined an electric dipole as consisting of two large equal and opposite charges a short distance apart.

In the olden days it was supposed that there were positive and negative **magnetic poles,** and that these acted on each other with inverse-square-law forces; so that two of them, a short distance apart, gave rise to a magnetic dipole with its magnetic moment **m**. Now the earth's magnetism may be regarded, approximately, as due to a large magnetic dipole pointing roughly along the polar axis. So isolated magnetic poles that were attracted to the North were called **North poles,** or North-seeking poles; and similarly for **South poles.** If we accept this postulate it is possible to develop all the analysis we require just as in Chapters II to IV; but the whole scheme is purely formal because we can never isolate a N. or S. pole. As Poisson discovered when breaking a permanent magnet, N. and S. poles always exist in pairs, forming in fact magnetic dipoles.

For this reason we shall not use the idea of an isolated pole in this book, except occasionally to illustrate a

phenomenon. It is not necessary to presume the existence of any magnetic poles, and it is better to discuss all magnetic phenomena in terms of magnetic dipoles. We say that a magnetic dipole of moment **m** equal to i **dS** is associated with a current i in a small coil of area **dS**. All the formulæ we require, such as field, torque, energy, etc., depend upon **m** and not upon the pole strength or distance apart.

§ 48. Large coils

Our discussion has so far centred around small coils. Suppose (Fig. 33) that a current i flows round a large coil having the contour s. Draw any surface S spanning s and divide it by a grilled network of lines into a series of small closed contours, such as $ABCD$, $AEFB$. If we imagine a current i to flow around each such contour in the direction marked by the arrows, the total current in AB will be zero, since the contributions from the two marked contours exactly cancel. The same is true for all lines except the outer contour s, in which a current i flows. Since no current flows in AB it does not matter whether there is a wire there or not.

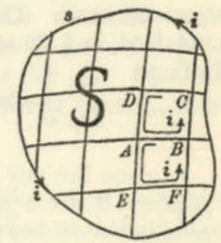


FIG. 33

Thus the current i in the large coil is equivalent to a current i in each of the smaller circuits. The magnetic effect of the large coil is therefore the same as that of a series of magnetic dipoles of moment i **dS** covering the surface S. This is referred to as a **magnetic shell** of strength i. The reader will recognise that it is the magnetic counterpart to the electric double layer discussed in § 20. If we were working in terms of magnetic poles we could regard a magnetic shell as a layer of N. poles slightly separated from a parallel layer of S. poles.

We must be careful, however, not to assign too real a character to this magnetic shell; for any surface that spans

the wire s is an adequate representation of the magnetic effect of the original current; and all such magnetic shells must be equivalent in calculating the potential.

Arguments similar to those of § **18** in Chapter III show that the magnetostatic potential Ω_P at P, due to a single magnetic dipole of moment $\mathbf{m}$ is

$$\Omega_P = \frac{\mathbf{m}.\mathbf{r}}{r^3} = \frac{m \cos \theta}{r^2}, \quad . \quad . \quad . \quad (6)$$

where θ is the angle between the direction of $\mathbf{m}$ and the radius vector from the dipole to P.

In the case of a magnetic shell, we have to replace $\mathbf{m}$ in (6) by $i\,\mathbf{dS}$, and then integrate over all elements $\mathbf{dS}$ of the magnetic shell. The calculation is just like that which was explained in § **20** when discussing the electric double layer. In fact:

$$\Omega_P = \int \frac{i\, dS \cos \theta}{r^2} = \int i\, d\omega = i\,\omega_P, \quad . \quad . \quad (7)$$

where ω_P is the solid angle subtended by the wire at P. The solid angle is to be regarded as positive if P lies on the positive side of the wire (see (iii) in § **44**). Equation (7) shows that the final value of Ω depends only on the current i, the shape of the wire and the position of P; as we should expect, no mention of which particular magnetic shell we choose occurs in the potential. This must be so, since as we said before, all magnetic shells are equivalent, provided only that they are bounded by the given wire s.

Formula (7) is important. Let us use it to calculate the change in Ω, written $[\Omega]$, when we move round a complete closed path. As the point P moves ω_P changes in a continuous manner, but if we return to our starting point without passing through s, then ω_P returns to its original value, so that $[\Omega] = 0$. If, however, our path embraces s once, there is a change of 4π in ω. The student should draw a few diagrams to illustrate this, and verify that if the path passes through

the wire circuit in the positive direction, ω decreases by 4π, so that

$$[\Omega] = -4\pi i.$$

We call this "threading the circuit positively." Thus, in general terms:

for a path not threading the coil, $[\Omega] = 0$,
for a path threading the coil N times, $[\Omega] = -4\pi N i$. (8)

If the coil consists of n turns of wire wound close together, the values given in (8) must all be multiplied by n.

The quantity $4\pi i$, which we introduced above, represents the drop in magnetostatic potential as we traverse a particular contour. It plays a somewhat similar role in magnetism to the corresponding drop in electrostatic potential which occurs in electrical problems, and which we refer to as electromotive force (e.m.f., represented by $\mathcal{E}$). For that reason we call it the **magnetomotive force** (m.m.f., represented by F), and write:

$F = 4\pi i$ for a closed path embracing the coil once,
$F = 4\pi N i$ for a closed path embracing the coil N times,
$F = 4\pi n i$ for a closed path embracing a coil of n turns once.

If we were using isolated magnetic poles, we should interpret this in language which is probably familiar to the reader, viz. "the work done in carrying a unit pole round a complete circuit is $4\pi i$, where i is the current embraced within the path."

Equation (8) shows that Ω is not a single-valued function, but may have any number of values all differing by an integral multiple of $4\pi i$. The student will wonder why, if the magnetostatic potential is many-valued, the corresponding electrostatic potential is not. The explanation lies in the fact that in the electrostatic case, as we cross a double layer of strength p there is a sudden change in V of $4\pi p$ (see § **20** and also § **23**, ques. 20). But in the case of the current, the magnetic shell is merely an artifice to enable us to calculate

the potential: as we have seen, it has no real existence (though it may have for permanent magnets, see Chapter VIII). Hence there is no corresponding jump in Ω to balance the decrease of 4π in solid angle.

We can write (8) in a useful form by using the fact that $\mathbf{H} = -\text{grad}\ \Omega$. For if **ds** is any small element of length

$$\mathbf{H}.\mathbf{ds} = -\text{grad}\ \Omega.\mathbf{ds} = -\frac{\partial\Omega}{\partial s}\,ds = -d\Omega,$$

where $d\Omega$ is the change in Ω along ds. Integrating this result for any closed path which threads the circuit once, we have

$$\int \mathbf{H}.\mathbf{ds} = -\int d\Omega = -[\Omega] = 4\pi i \quad . \quad (9)$$

If we cross the magnetic shell from the negative to the positive side, then the potential Ω decreases by $4\pi i$ in the complete circuit: this verifies that our sign in (9) is correct. If we take a closed path which does not embrace the current, then (9) becomes

$$\int \mathbf{H}.\mathbf{ds} = 0. \quad . \quad . \quad . \quad . \quad (10)$$

§ 49. Solenoids and circular coils

Consider a circular wire of radius a, with its centre at O, carrying a current i as shown in Fig. 34. The potential at P, by (7), is

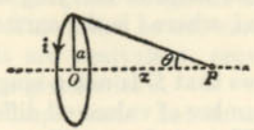


FIG. 34

$$\begin{aligned}\Omega_P &= i\omega_P\\ &= 2\pi i(1-\cos\theta)\\ &= 2\pi i\left(1-\frac{z}{\sqrt{(a^2+z^2)}}\right), \quad . \quad . \quad . \quad (11)\end{aligned}$$

where $z = OP$.

From this we soon calculate the field **H** at points on the axis. (A full discussion of this problem enabling us to calculate Ω at points off the axis as well, is reserved for Chapter IX, § 81). In fact

$$H = -\frac{\partial\Omega}{\partial z} = \frac{2\pi i a^2}{(a^2+z^2)^{3/2}} \quad . \quad . \quad . \quad (12)$$

At the centre of the coil, $H = 2\pi i/a$. If there are n turns we obtain the familiar formula used in tangent galvanometers

$$H = 2\pi n i/a. \quad . \quad . \quad . \quad (13)$$

Next suppose that we lay a large number of similar coils side by side with their centres along OP, so that they form a **solenoid.** We can effectively achieve this by winding the wire in a helical shape. If each turn carries the same current i and there are n turns per unit length, then the total potential at P is found by integrating the previous formula for Ω_P. In fact, if the ends of the coil are given by $\theta = \alpha$, $\theta = \beta$ (see Fig. 35), and z is measured in the direction OP,

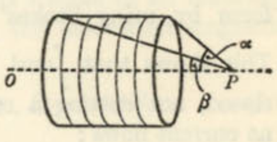


FIG. 35

$$\begin{aligned}\Omega_P &= \int_{\theta=\beta}^{\theta=\alpha} 2\pi i\,(1-\cos\theta)\,n\,dz\\ &= -2\pi n i\int_{\alpha}^{\beta}(1-\cos\theta)a\,\text{cosec}^2\,\theta\,d\theta\\ &= 2\pi n i a\,(\tan\tfrac{1}{2}\alpha-\tan\tfrac{1}{2}\beta) \quad . \quad . \quad (14)\end{aligned}$$

We can calculate the field **H** at P either by a corresponding integration of (12), or, more easily, by using (14) in the equation $\mathbf{H} = -\text{grad}\ \Omega$. By symmetry **H** is directed along the axis of the solenoid, and in fact

$$H_P = 2\pi n i\,(\cos\beta - \cos\alpha) \quad . \quad . \quad . \quad . \quad (15)$$

If the solenoid is very long, then at all points inside it, but

some distance from the ends, we can put $\alpha = +\pi$, $\beta = 0$. Thus the field inside a long solenoid is

$$H = 4\pi ni \quad . \quad . \quad . \quad . \quad (16)$$

§ 50. Flux theorem for H

Since there are no isolated magnetic poles, it follows that lines and tubes of **H** can never end. The corresponding flux theorem (see § 9) is that the flux of **H** out of any volume is zero, i.e. $\int \mathbf{H}.d\mathbf{S} = 0$. This leads at once to the differential form

$$\text{div}\,\mathbf{H} = 0 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

We can also write the circuital equation (10) in a differential form by using Stokes' theorem that $\int \mathbf{a}.d\mathbf{s} = \int \text{curl}\,\mathbf{a}.d\mathbf{S}$. This shows that $\int \text{curl}\,\mathbf{H}.d\mathbf{S} = 0$ for any surface, open or closed, not cutting a current. Hence in any region where no current flows :

$$\text{curl}\,\mathbf{H} = 0 \quad . \quad . \quad . \quad . \quad (18)$$

Now since curl $\mathbf{H} = 0$ it follows that **H** may be expressed as the gradient of a scalar quantity. This is of course the magnetostatic potential Ω, which justifies our assertion in § 46 that $\mathbf{H} = -\text{grad}\,\Omega$.

If we combine div $\mathbf{H} = 0$ and $\mathbf{H} = -\text{grad}\,\Omega$, we once more obtain Laplace's equation

$$\nabla^2\Omega = 0 \quad . \quad . \quad . \quad . \quad (19)$$

It is important to realise that this only holds in regions not occupied by current.

§ 51. Field of a straight wire

By taking any solution of (19) we can obtain the magnetic field for a certain set of currents. Now one of the simplest solutions in cylindrical polar co-ordinates r, θ, z is

$$\Omega = A\theta,$$

where A is constant. It follows,* since $\mathbf{H} = -\text{grad}\,\Omega$, that

$$H_r = 0,\ H_\theta = -\frac{A}{r},\ H_z = 0.$$

If we take any closed path which does not embrace the z axis, $[\Omega] = 0$. Hence, by (8) the only current must be along this axis. If our path embraces the z axis once positively the increase of Ω in going round it is $2\pi A$, so that $[\Omega] = 2\pi A$. Thus, again using (8), we see that the current flowing along the positive direction of the z axis is $-A/2$, a result that could equally well have been obtained according to (9) by integrating $\mathbf{H}.d\mathbf{s}$ round a circle parallel to the xy plane whose centre was on the z axis.

This has proved that if a current i flows in an infinitely long wire (taken to be the z axis), the associated magnetic field is

$$H_r = 0\ ,\ H_\theta = \frac{2i}{r}\ ,\ H_z = 0\ ,\ \Omega = -2i\theta \quad . \quad . \quad (20)$$

Lines of **H** are therefore circles with centres on the wire, in planes perpendicular to the current. The reader has probably seen diagrams in which this fact is shown experimentally by means of iron filings which tend to align themselves in such circles.

§ 52. Volume currents and the Vector Potential

We have seen in § 50 that the equation curl $\mathbf{H} = 0$ breaks down in a region where currents are flowing. To see what replaces it, we suppose that the current is specified by the current vector $\mathbf{j}$. According to (9) $\int \mathbf{H}.d\mathbf{s}$ round any closed contour s is equal to 4π times the total current which is embraced by the contour. Let us take any surface S which spans s. Then the current embraced by the contour is equal

* Formulæ for grad and curl in cylindrical polars are in Rutherford's *Vector Methods*, 1946, pp. 72 and 73.

to the total current crossing S; that is, the flux of $\mathbf{j}$ across S (see § 33, equation 2). So

$$\int \mathbf{H}.d\mathbf{s} = 4\pi \int \mathbf{j}.d\mathbf{S} \quad . \quad . \quad . \tag{21}$$

Transforming by Stokes' theorem:

$$\int (\text{curl } \mathbf{H} - 4\pi\mathbf{j}).d\mathbf{S} = 0.$$

This is true for all surfaces S. Hence the integrand is identically zero, and the required new equation is

$$\text{curl } \mathbf{H} = 4\pi\mathbf{j}. \quad . \quad . \quad . \tag{22}$$

We can soon see that it makes no difference which surface S we take to span s. In fact, if there are two surfaces S_1 and S_2 spanning s, together they form a closed surface, and the total flux of $\mathbf{j}$ out of this surface is zero (equation (5) of § 33, with $\dfrac{\partial\rho}{\partial t} = 0$). Bearing in mind the directions of the normals to $d\mathbf{S}_1$ and $d\mathbf{S}_2$, this gives $\int \mathbf{j}.d\mathbf{S}_1 = \int \mathbf{j}.d\mathbf{S}_2$, proving that (21) is independent of the particular surface S chosen. Now curl grad $\Omega \equiv 0$, so that equation (22) shows that in regions occupied by currents ($\mathbf{j} \neq 0$) we can no longer write $\mathbf{H}$ as $-$grad Ω. In such cases the idea of a magnetostatic potential has to be abandoned. We therefore introduce a new type of potential, which becomes increasingly important in more advanced work.

Since the equation div $\mathbf{H} = 0$ is not modified by the presence of $\mathbf{j}$, it follows that we may write $\mathbf{H}$ as the curl of a vector. So let

$$\mathbf{H} = \text{curl } \mathbf{A}^*. \quad . \quad . \quad . \tag{23}$$

$\mathbf{A}$ is not completely defined by this equation since, if ψ is any scalar function, curl $(\mathbf{A} + \text{grad } \psi) =$ curl $\mathbf{A}$. To complete the definition we usually add the extra condition that

$$\text{div } \mathbf{A} = 0 \quad . \quad . \quad . \quad . \tag{24}$$

* In the next chapter (23) is replaced by the more general equation $\mathbf{B} = \text{curl } \mathbf{A}$.

The vector $\mathbf{A}$ so defined is called the **magnetic vector potential**, or sometimes simply the **vector potential**.

Now curl curl$\equiv$grad div $-\nabla^2$, so by combining (22)-(24) we obtain the standard differential equation for $\mathbf{A}$:

$$\nabla^2\mathbf{A} = -4\pi\mathbf{j} \quad . \quad . \quad . \quad . \tag{25}$$

This is a vector equation, each component of which resembles Poisson's equation (§ 10, eq. 20), and may be solved in the same way. In fact, provided that $\mathbf{A}$ tends to zero at infinity and there are no surface currents (or current sheets, as they are usually called), we may use the analysis of §§ 7 and 13 to show that the appropriate solution of (25) is

$$\mathbf{A} = \int \frac{\mathbf{j}\,dv}{r} \quad . \quad . \quad . \quad . \tag{26}$$

It is possible to show, though we shall not do so here, that this satisfies the auxiliary condition (24), viz. div $\mathbf{A} = 0$.

(26) takes a simple form if the current flows in a series of thin filaments, or wires. For then $\mathbf{j}$ is everywhere in the direction of the wire and so $\mathbf{j}\,dv$ may be replaced by $i\,d\mathbf{s}$, where i is the current flowing in the wire and $d\mathbf{s}$ is the vector element of length along the wire. Thus for a wire carrying current i

$$\mathbf{A} = i \int \frac{d\mathbf{s}}{r}. \quad . \quad . \quad . \quad . \tag{27}$$

Each element of wire may be said to make a contribution $(i/r)d\mathbf{s}$ to $\mathbf{A}$. Now

$$\text{curl } \frac{i\,d\mathbf{s}}{r} = \frac{i}{r} \text{ curl } d\mathbf{s} + \left(\text{grad } \frac{i}{r}\right) \times d\mathbf{s}.$$

The differentiations in this equation are with respect to the co-ordinates of the point P at which the field is being determined. These co-ordinates do not appear in $d\mathbf{s}$ and so curl $d\mathbf{s} = 0$. Also, if $\mathbf{r}$ is measured from $d\mathbf{s}$ to P,

$\operatorname{grad}_P \frac{i}{r} = -\frac{i}{r^3}\mathbf{r}$. Hence the contribution to $\mathbf{H}$ from the element $\mathbf{ds}$ is $\frac{i}{r^3}\mathbf{ds}\times\mathbf{r}$, and by summation along the whole wire,

$$\mathbf{H} = i\int\frac{\mathbf{ds}\times\mathbf{r}}{r^3}. \quad . \quad . \quad . \quad (28)$$

In the special case of an infinite straight wire, in terms of the notation of Fig. 36,

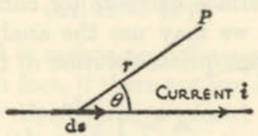


FIG. 36

$$H = i\int\frac{\sin\theta\, ds}{r^2} \quad . \quad . \quad . \quad (29)$$

This equation provides the easiest way of calculating the field of a uniform straight current, which we have already found by other means in § **51**.

Equations (28) and (29) are known as the **Biot-Savart law.**

§ 53. Coefficient of mutual induction

We have seen in (4) that the potential energy of a small coil of area $\mathbf{dS}$ carrying constant current i in a magnetic field $\mathbf{H}$ is $-i\,\mathbf{H}.\mathbf{dS}$. If we have a large coil we can replace it by a series of small ones, as in § **48**, and the potential energy is therefore

$$-i\int\mathbf{H}.\mathbf{dS}\,. \quad . \quad . \quad (30)$$

An important case arises when the field $\mathbf{H}$ is itself created by a current in some other coil. If we call the first current i_1 and the second i_2, this potential energy is

$$-i_1\int\mathbf{H}_2.\mathbf{dS}_1 \quad . \quad . \quad (31)$$

Putting $\mathbf{H}_2 = \operatorname{curl}\mathbf{A}_2$ as in (23), and then using Stokes' theorem, we have

$$\int\mathbf{H}_2.\mathbf{dS}_1 = \int\operatorname{curl}\mathbf{A}_2.\mathbf{dS}_1 = \int\mathbf{A}_2.\mathbf{ds}_1,$$

in which the integration is along the contour of the current i_1. Now by (27) $\mathbf{A}_2 = i_2\int\frac{\mathbf{ds}_2}{r}$, so that the mutual potential energy of two coils carrying invariable currents i_1 and i_2 is

$$-M_{12}i_1i_2, \quad . \quad . \quad . \quad (32)$$

where M_{12}, which is known as the **coefficient of mutual induction** of the two circuits, is given by **Neumann's formula**:

$$M_{12} = \int\frac{\mathbf{ds}_1.\mathbf{ds}_2}{r} \quad . \quad . \quad (33)$$

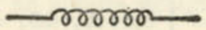
r is the distance between the two vector elements $\mathbf{ds}_1$, $\mathbf{ds}_2$ of the two coils and the integration is over the whole of s_1 and s_2. Comparison of (31) and (32) shows that M_{12}* is the flux of $\mathbf{H}$ through circuit 1 when unit current flows in circuit 2: and the symmetry of (33) shows also that this is the same as the flux of $\mathbf{H}$ through circuit 2 when unit current flows in circuit 1, i.e. $M_{12} = M_{21}$.

If M_{12} is calculated it becomes possible to calculate the forces exerted on either coil by the other, assuming always that their currents are kept fixed. Examples of this will be found at the end of the chapter.

* In the presence of magnetisable matter (Chapter VII) M and L are defined in terms of the flux of $\mathbf{B}$ rather than of $\mathbf{H}$. (See § **58**.)

§ 54. **Coefficient of self-induction**

In Chapter XI we shall require a quantity closely related to M_{12}. Thus, if we make the circuits 1 and 2 in § 53 coincide, we obtain a quantity M_{11}, generally written L, and called the self-induction of the circuit. Li represents the flux of **H** * through the circuit when current i flows in it. The usual symbol for L is a coiled line



L

FIG. 36A

At first sight it would seem that we could calculate L by using Neumann's formula (33) with $\mathbf{ds}_1$ and $\mathbf{ds}_2$ elements of the same circuit. But a simple calculation shows that this gives a logarithmic infinity, showing us that the formula has broken down. This is because we have regarded the current as flowing in an infinitely thin filament. To obtain a valid formula we have to take account of the shape of any section of the wire. The actual calculation of L which, like M, is a purely geometrical property of the circuit, is somewhat involved, and we shall not discuss it further here. [See, however, § 116, questions 16-18.]

If there is more than one circuit, the principle of superposition tells us that the flux of **H** through any one coil is the sum of contributions from each of the currents separately. Thus for the case of two circuits with currents i_1 and i_2, by combining the results of this and the previous paragraph, we see that the flux through circuit 1 is $L_1i_1+M_{12}i_2$; and similarly the flux through circuit 2 is $L_2i_2+M_{12}i_1$.

§ 55. Examples

1. Obtain the field in a solenoid by differentiation of the potential (14).

2. Obtain the formula (12) for the field at points on the axis of a circular loop by direct application of the Biot-Savart law (28).

3. Two equal circular coils of radius a are placed opposite one another, a distance $2b$ apart. They carry the same current in the same direction. Show that at the point midway between the two centres the first three differential coefficients of the field are zero if $2b = a$. Such an arrangement which gives an approximately uniform field is known as a **Helmholtz Coil.**

4. An infinite straight wire whose cross-section is a circle of radius a carries a uniform current i. Use equations (17) and (21) to show that the magnetic field is given by :

$$r>a,\ H_r = 0,\ H_\theta = 2i/r,\quad H_z = 0,$$
$$r<a,\ H_r = 0,\ H_\theta = 2ir/a^2,\ H_z = 0.$$

Notice that this amplifies the formulæ in (20) where we supposed that the wire was infinitely thin.

5. Starting from equation (26) prove, after the manner of (28), that with a current vector $\mathbf{j}$ each element of volume dv contributes to the magnetic field an amount

$$\left[\frac{\mathbf{j}\times\mathbf{r}}{r^3}\right] dv.$$

6. Show that the magnetic field at the centre of a square coil of side $2a$ carrying a current i is $(4i\sqrt{2})/a$.

7. A current i flows in an infinite thin wire coincident with the z axis. The return flow takes place along a parallel wire through the point $(R, 0, 0)$ where R is very large. Show that the magnetostatic potential at a point whose cylindrical coordinates are (r, θ, z) is $\Omega = 2i(\pi-\theta)$. Deduce that the magnetic field is (cf. (20))

$$H_r = 0,\ H_\theta = 2i/r,\ H_z = 0.$$

8. Show that the formula (27) gives an infinite value for the vector potential of an infinite straight current. But if we suppose that the current returns along a parallel wire, then at a point P the vector potential has magnitude $2i \log (b/a)$, where a and b are the shortest distances from P to the two wires. This is known as a **bifilar current.**

9. Obtain the value of the magnetic field for a current i in an infinite straight wire (§ 51) by using the Biot-Savart law (29).

* In the presence of magnetisable matter (Chapter VII) M and L are defined in terms of the flux of **B** rather than of **H**. (See § 58.)

10. In the case of an infinite straight wire carrying a current i, equation (27) for the vector potential breaks down since $\mathbf{A}$ does not tend to zero at infinity. But verify that a correct form for $\mathbf{A}$ in such a case is $\mathbf{A} = (-2i \log r)\,\mathbf{a}$ where $\mathbf{a}$ is a unit vector in the direction of the current and r is the perpendicular distance from the wire.

11. A current i flows in a circular wire $r = a$, $z = 0$ in cylindrical co-ordinates. Show that the vector potential at any point (r, θ, z) is given by

$$A_r = A_z = 0 \ , \ A_\theta = 2ia \int_0^\pi \frac{\cos\theta \, d\theta}{(a^2+r^2+z^2-2ar\cos\theta)^{\frac{1}{2}}}.$$

If the loop is small show that this is approximately $\pi i a^2 r/(r^2+z^2)^{\frac{3}{2}}$. Verify that this is the same as the result in question 12.

12. Verify that the vector potential at P due to a small magnetic dipole is $\mathbf{A} = -\mathbf{m}\times\operatorname{grad}_P\left(\frac{1}{r}\right) = \frac{\mathbf{m}\times\mathbf{r}}{r^3}$, where r is measured from the dipole to P.

13. Verify that the vector potential of a constant field H in the x direction may be written

$$A_x = 0, \quad A_y = -aHz, \quad A_z = (1-a)\,Hy,$$

where a is an arbitrary constant.

Similarly verify that for a constant field H parallel to the θ axis in spherical polar co-ordinates, we can write

$$A_r = 0 \ , \ A_\theta = 0 \ , \ A_\phi = \tfrac{1}{2} Hr \sin\theta.$$

Prove that in cylindrical co-ordinates, if H is parallel to the z axis,

$$A_r = 0 \ , \ A_z = 0 \ , \ A_\theta = \tfrac{1}{2}Hr.$$

14. A coil of wire of m turns is closely wound on a circular ring of any form of cross-section, and a second coil of n turns is intertwined with the first. Show that the coefficients of mutual and self-induction are

$$M_{12} = 2mn \int \frac{dS}{\rho} \ , \ L_1 = 2m^2 \int \frac{dS}{\rho} \ , \ L_2 = 2n^2 \int \frac{dS}{\rho},$$

where ρ is the distance of any point in the cross-section from the axis of the ring. Deduce that $L_1L_2 = M_{12}^2$, an important relation

in the theory of transformers (see § 96). In practice, on account of leakage, $M_{12}^2 < L_1L_2$, and the ratio $M_{12}/\sqrt{(L_1L_2)}$ is called the **Coupling Factor.**

15. Two equal circular loops of radius a lie opposite each other, a distance c apart. Show that the coefficient of mutual induction is

$$M_{12} = 2\pi a^2 \int_0^{2\pi} \frac{\cos\psi \, d\psi}{(c^2+2a^2-2a^2\cos\psi)^{\frac{1}{2}}}.$$

If c is very large, show that $M_{12} = 2\pi^2 a^4/c^3$. Deduce that if unit currents flow in the same direction round the coils, they attract each other with a force $6\pi^2a^4/c^4$. (See also Chapter IX, question 19.)

16. Show that if we may assume that each moving charge in equations (27), (28) contributes independently of all other charges, then a charge e moving in free space with velocity $\mathbf{v}$ creates a vector potential $\mathbf{A}$ and a magnetic field $\mathbf{H}$ given by

$$\mathbf{A} = \frac{e\mathbf{v}}{r} \ , \ \mathbf{H} = e\,\frac{\mathbf{v}\times\mathbf{r}}{r^3}.$$

Deduce that if $\mathbf{E}$ is the electrostatic field due to the charge, $\mathbf{H} = \mathbf{v}\times\mathbf{E}$. (These formulæ are only true if the velocity $\mathbf{v}$ is much less than the velocity of light.)

CHAPTER VII

STEADY CURRENTS IN MAGNETIC MATERIAL

§ 56. Magnetic media

In Chapter I we described Faraday's suggestion that each atom contained within itself tiny electric currents—a hypothesis which has been amply confirmed by modern atomic theory. It follows that in the presence of a magnetic field **H** these atoms will experience couples tending to orient them in the direction of **H**, the only exceptions being atoms in which the several minute currents cancel each other. This is Weber's hypothesis to explain how a medium may become magnetised by an imposed field. As a result of this orientation there is an effective magnetic moment in the direction of **H**. Exactly the same situation occurred in Chapter IV, where we saw that an electrostatic field created an effective electric dipole moment in the medium. We described that situation in terms of a polarisation vector **P**. The corresponding vector in the magnetic case is called the **intensity of magnetisation I.** Its definition is that each element of volume dv behaves as if it were a magnetic dipole or small magnet of moment $\mathbf{I}\,dv$. The fact that we introduce **I** in a precisely similar manner to **P** will enable us to apply a great deal of our previous analysis to the present problem of steady currents in magnetic materials.

The intensity of magnetisation **I** is a vector quantity, which except in certain crystalline materials will be in the same direction as **H**. In fact, apart from certain ferromagnetic substances (e.g. iron, Heusler alloys, etc.), **I** is proportional to **H** so that we may usually write

$$\mathbf{I} = \kappa \mathbf{H}, \quad . \quad . \quad . \quad . \quad (1)$$

where κ is the magnetic susceptibility of the material. There is one difference between this equation and the corresponding electrostatic equation $\mathbf{P} = k\mathbf{E}$. The susceptibility may be either negative or positive. If $\kappa < 0$ we describe the material as diamagnetic; if $\kappa > 0$, we say that it is paramagnetic. Typical values of κ are (using c.g.s.e.m.u.) $\kappa = -3\times10^{-6}$ for a diamagnetic medium such as gold: $\kappa = 30\times10^{-6}$ for a paramagnetic medium such as platinum. This latter value is at room temperature: **Curie's law** states that for paramagnetic materials κ is proportional to $1/T$, where T is the absolute temperature. If the field is not too big certain soft irons obey the law (1) with κ somewhere between 50 and 1000, but as **H** increases κ becomes much smaller in such a way that **I** increases to an asymptotic value called the **saturation magnetisation.**

In what follows we shall not require to know just why κ is positive or negative: this indeed is the realm of quantum theory. It will be sufficient to apply (1) treating κ as a known constant.

§ 57. Magnetic induction

The magnetisation **I** will give rise to a magnetostatic potential which we can calculate just as we calculated the potential due to a polarisation **P** in § 25 of Chapter IV. For the potential due to a magnetic dipole **m** is $\mathbf{m}.\mathrm{grad}\,\frac{1}{r}$. So, putting $\mathbf{m} = \mathbf{I}\,dv$ and integrating over any volume v, we see that the potential due to **I** is $\int \mathbf{I}.\mathrm{grad}\,\frac{1}{r}\,dv$. This may be transformed, just as in § 25 by the use of Green's theorem, and becomes

$$\int \frac{\mathbf{I}.d\mathbf{S}}{r} - \int \frac{\mathrm{div}\,\mathbf{I}}{r}\,dv \quad . \quad . \quad . \quad (2)$$

The interpretation of this is that the effect of the magnetisation **I** in a certain volume v is precisely the same as if we

had a volume source of magnetic poles $-\text{div}\ \mathbf{I}$ and a boundary layer I_n. These are sometimes known as **Poisson's imaginary magnetic matter,** and are evidently strictly parallel to the "apparent charges" introduced in § 25. The flux theorem for $\mathbf{H}$ (§ 50, equation 17) now becomes

$$\text{div}\ \mathbf{H} = 4\pi \times \text{volume density of magnetic poles} = -4\pi\ \text{div}\ \mathbf{I}.$$

Let us introduce a vector $\mathbf{B}$, called the **magnetic induction,** defined by the relation

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Then the flux theorem is

$$\text{div}\ \mathbf{B} = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The reader will notice how closely the magnetic induction vector $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I}$ corresponds with Maxwell's displacement vector $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ (§ 26). The chief difference is that since there are no isolated true magnetic poles div $\mathbf{B}$ is always zero, whereas div $\mathbf{D} = 4\pi\rho$.

This last fact means that there are no sources of $\mathbf{B}$, so that the lines and tubes of $\mathbf{B}$ do not end. We have already seen (Chapter VI) that in free space this is true for $\mathbf{H}$. Now we notice that in free space $\kappa = 0$, so that $\mathbf{I} = 0$ and $\mathbf{B} = \mathbf{H}$. Thus the results of Chapter VI are included as special cases of the two equations (3) and (4).

In homogeneous isotropic materials where $\mathbf{I} = \kappa\mathbf{H}$, it follows from (3) that

$$\mathbf{B} = \mu\mathbf{H}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where

$$\mu = 1 + 4\pi\kappa \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This quantity μ is called the **magnetic permeability.** In free space $\mu = 1$; in diamagnetic materials $\mu < 1$; in paramagnetic materials $\mu > 1$; in ferromagnetics μ is very large.

Before we can solve problems we must discuss the continuity conditions satisfied by $\mathbf{B}$ and $\mathbf{H}$ at the boundary between two media in which the permeability is different (Fig. 37). The parallelism between the magnetic and electric

cases enables us to argue just as in (ix) of § 26, with the conclusion

$$B_{n_1} = B_{n_2},\ H_{s1} = H_{s_2} \quad . \quad . \quad . \quad (7)$$

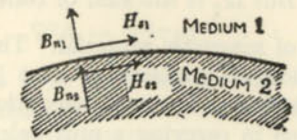


FIG. 37

If, however, there is a surface current flowing along the boundary the last equation requires alteration. This does not often occur and so we shall not discuss it here (see § 65, question 2).

§ 58. Equations of the magnetic field

We are now able to write down all the necessary equations of the magnetic field. Thus:

(i) $$\text{div}\ \mathbf{B} = 0 \quad . \quad . \quad . \quad . \quad (4)$$

(ii) $$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I}, \quad . \quad . \quad . \quad . \quad (3)$$

or, in most cases

$$\mathbf{B} = \mu\mathbf{H} \quad . \quad . \quad . \quad . \quad (5)$$

(iii) If the current vector is $\mathbf{j}$, then just as in (22) of § 52,

$$\text{curl}\ \mathbf{H} = 4\pi\mathbf{j} \quad . \quad . \quad . \quad . \quad (8)$$

On account of the induced magnetisation $\mathbf{I}$ this latter equation requires some justification. We start with the relation (9) of § 48, viz., that since $\mathbf{H} = -\text{grad}\ \Omega$, therefore for any closed curve s, $\int \mathbf{H}.d\mathbf{s} = -[\Omega]$. However, Ω is itself the sum of two parts. Let us write $\Omega = \Omega_1 + \Omega_2$, where Ω_1 arises from a series of currents i flowing in closed filaments, and Ω_2 arises from the induced magnetisation $\mathbf{I}$. According to (7) of § 48, at any point P, $\Omega_1 = \Sigma i\omega_P$, where

ω_P is the solid angle subtended at P by the circuit carrying current i. Thus $[\Omega_1] = -4\pi\Sigma i$, where Σi is the total current which threads the curve s. So far our argument is the same as in Chapter VI. But Ω_2 is the sum of contributions $\dfrac{\mathbf{I}dv\,.\,\mathbf{r}}{r^3}$ from each element of magnetic material. This quantity is a single-valued function of position, so that $[\Omega_2] = 0$. This result may be interpreted in terms of magnetic poles by saying that no work is done in carrying a unit pole round a closed curve in the presence of a series of constant magnets. Combining the two parts of Ω,

$$\int \mathbf{H}.\mathbf{ds} = -[\Omega_1+\Omega_2] = 4\pi\times\text{total current threading } s.$$

From here it follows exactly as in § 52 that in the presence of a volume current $\mathbf{j}$,

$$\text{curl } \mathbf{H} = 4\pi\mathbf{j}.$$

(iv) At places where there is no current we can express $\mathbf{H}$ in terms of a magnetostatic potential,

$$\mathbf{H} = -\text{grad } \Omega, \quad . \quad . \quad . \quad (9)$$

so that

$$\text{div } (\mu \text{ grad } \Omega) = 0 \quad . \quad . \quad . \quad (10)$$

If μ is constant, as usually happens,

$$\nabla^2\Omega = 0 \quad . \quad . \quad . \quad . \quad (11)$$

(v) At places where there is a current, no magnetostatic potential exists, and as in § 52 we introduce a magnetic vector potential $\mathbf{A}$. In fact, since div $\mathbf{B} = 0$ at all points whether or not there is a current flowing there, we must now write

$$\mathbf{B} = \text{curl } \mathbf{A}, \quad . \quad . \quad . \quad . \quad (12)$$

with the condition, similar to (24) of § 52

$$\text{div } \mathbf{A} = 0 \quad . \quad . \quad . \quad . \quad (13)$$

Taking the curl of (12), and using (5) and (8), we have

$$\text{curl}\left(\frac{1}{\mu}\text{ curl } \mathbf{A}\right) = \text{curl } \mathbf{H} = 4\pi\mathbf{j} \quad . \quad . \quad . \quad (14)$$

If μ is constant this becomes

$$\text{curl curl } \mathbf{A} = 4\pi\mu\mathbf{j}.$$

But curl curl$\equiv$grad div $-\nabla^2$, and div $\mathbf{A} = 0$, so that the equation for $\mathbf{A}$ is

$$\nabla^2\mathbf{A} = -4\pi\mu\mathbf{j} \quad . \quad . \quad . \quad . \quad (15)$$

In free space $\mu = 1$, so that this reduces at once to (25) of § 52. Provided that $\mathbf{A}$ tends to zero at infinity and there are no current sheets its solution is

$$\mathbf{A} = \mu\int\frac{\mathbf{j}\,dv}{r}. \quad . \quad . \quad . \quad (16)$$

If the field is due to a current i in a thin wire we may replace $\mathbf{j}\,dv$ by i $\mathbf{ds}$, where $\mathbf{ds}$ is an element of length of the wire, in which case

$$\mathbf{A} = \mu i\int\frac{\mathbf{ds}}{r}. \quad . \quad . \quad . \quad (17)$$

Since $\mathbf{B}$ = curl $\mathbf{A}$ it follows, just as in § 52, that

$$\mathbf{B} = \mu i\int\frac{\mathbf{ds}\times\mathbf{r}}{r^3} \quad . \quad . \quad . \quad (18)$$

This means that the Biot-Savart law (28) of § 52 still holds:

$$\mathbf{H} = i\int\frac{\mathbf{ds}\times\mathbf{r}}{r^3} \quad . \quad . \quad . \quad (19)$$

As a consequence of (19) all the results proved in Chapter VI for the magnetic field due to a straight wire, or a loop, or a solenoid, are unaffected by the presence of a uniform medium. We do not need to repeat the calculations.

(vi) At a sudden change of medium

$$B_n \text{ and } H_s \text{ are continuous.} \quad . \quad . \quad . \quad (7)$$

We ought also to discuss the boundary conditions for $\mathbf{A}$ at a change of medium. These are rather more involved than the earlier type of boundary condition, and we shall be content

therefore to quote the answer. Calling the media 1 and 2 as in Fig. 37, and supposing that the intensity of magnetisation is $\mathbf{I}_1$ and $\mathbf{I}_2$ in the two media, these are

$$\mathbf{A}_1 = \mathbf{A}_2 \qquad . \quad . \quad . \quad . \quad . \quad (20)$$

$$\frac{\partial \mathbf{A}_1}{\partial n} - \frac{\partial \mathbf{A}_2}{\partial n} = 4\pi[(\mathbf{I}_1 - \mathbf{I}_2) \times \mathbf{n}], \qquad . \quad . \quad (21)$$

where $\mathbf{n}$ is a unit vector directed along the normal from 1 to 2. It is possible to verify the sufficiency of these conditions by deducing from them the known conditions (7).

§ 59. Cavities

The boundary conditions (7) allow us to discuss the field inside a small cavity scooped out of the magnetic medium. Consideration of this problem is important because of the indefiniteness which attaches to our definition and measurement of **B** and **H** at points inside a medium. Our best plan is to suppose part of the medium around some point P

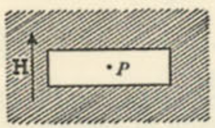


FIG. 38A

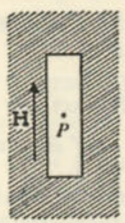


FIG. 38B

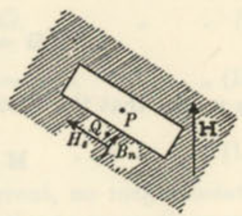


FIG. 38C

to be removed and then to determine **B** and **H** at P, inside the cavity. The argument is very similar to that already used in § 31. In fact, since the boundary conditions for **B** and **H** are exactly similar to those for **D** and **E**, the analysis of § 31 can be applied to this problem just as it stands. Thus, with a disc-shaped cavity, as in Fig. 38A, which may be compared with Fig. 26A, the magnetic field at P inside the hollow is the same as the induction **B** outside, and with

a needle-shaped cavity pointing in the direction of **H**, as in Fig. 38B, which compares with Fig. 26B, the magnetic field at P inside the cavity is the same as the field **H** outside.

We shall be concerned shortly with a rather more complicated case—a disc-shaped cavity at an oblique angle to **B** (Fig. 38C). Let us define the field at a point Q just outside the cavity in terms of the normal component B_n and the tangential component H_s. We choose these because they are the only components that do not change abruptly on crossing from Q to P. So at P, inside the cavity, the magnetic field **H** is compounded of B_n normal to the plane of the disc, and H_s along the plane.

§ 60. Potential energy of a small coil

Let us suppose that a small coil of area dS carrying a current i is imbedded in magnetic material at a point where there is a field defined by the vectors **B** and **H**. In order to place it there and then calculate its potential energy, we imagine the material of the medium to be removed from a small volume just surrounding the coil. This cavity will be of the disc-shaped type shown in Fig. 38C. Now according to (4) of § 46 the potential energy of the coil is given by $-i\mathbf{H}.d\mathbf{S}$, where **H** is the field inside the cavity. But we have just seen that inside the cavity the field is a combination of H_s and B_n; H_s is perpendicular to $d\mathbf{S}$ and B_n is parallel to $d\mathbf{S}$. So the required potential energy is $-iB_n dS$, or in vector form, simply:

$$\text{potential energy of coil} = -i\mathbf{B}.d\mathbf{S} \qquad . \quad . \quad (22)$$

We may write this as $-i$ times the flux of **B** through the coil. In free space, where $\mathbf{B} = \mathbf{H}$, we do not need to distinguish between the flux of **B** and of **H**. But (22) shows that in magnetic media it is **B** rather than **H** which gives the potential energy. For this reason some writers * define **B** in terms of the torque experienced by a small coil in a magnetic medium.

* E.g. Smyth, *Static and Dynamic Electricity*, 1939.

We have preferred to introduce **B** by the same type of reasoning that led to **D** in the electrostatic case, because this preserves the parallelism between the two types of field and is more directly related to known properties of the atom; further, the measurement of torque on a coil inside a solid body such as iron does present certain preliminary difficulties.

We can derive an alternative form for the energy (22) by putting $\mathbf{B} = \text{curl } \mathbf{A}$. By Stokes' theorem we can then write

$$-i\mathbf{B}.d\mathbf{S} = -i \text{ curl } \mathbf{A}.d\mathbf{S} = -i\int \mathbf{A}.d\mathbf{s}, \quad . \ (23)$$

where the integral is taken round the contour of the small loop. $\int \mathbf{A}.d\mathbf{s}$ is sometimes called the circulation of **A** round the loop. Since a larger loop may be regarded (§ 48) as the superposition of a series of smaller ones all carrying the same current, (23) shows that the potential energy of a larger coil carrying an invariable current i in the presence of an external field defined by its vector potential **A** is also $-i\int \mathbf{A}.d\mathbf{s}$, but now the integration is round the contour of the large coil. It is important to remember, for this purpose, that it is only that part of **A** which arises from the external field, and not that from the coil itself, which determines the mutual potential energy. The formula corresponding to (23) for the case of a volume distribution of current is found by replacing i **ds** by **j** dv, and it is

$$\text{potential energy of current system} = -\int \mathbf{A}.\mathbf{j}\, dv \quad . \ (24)$$

A particularly important form of (23) occurs when the field represented by **A** is itself due to a current in some other coil. Calling the first current i_1 and this second current i_2, the mutual potential energy is

$$-i_1\int \mathbf{A}_2.d\mathbf{s}_1.$$

But from (17):

$$\mathbf{A}_2 = \mu i_2\int\frac{d\mathbf{s}_2}{r},$$

so that the mutual potential energy is

$$-\mu i_1 i_2\int\frac{d\mathbf{s}_1.d\mathbf{s}_2}{r}.$$

Just as in (32) of § 53 we may write this in the form

$$\text{mutual potential energy} = -M_{12}i_1i_2,$$

where
$$M_{12} = \mu\int\frac{d\mathbf{s}_1.d\mathbf{s}_2}{r} \quad . \quad . \quad . \ (25)$$

Thus M_{12} is μ times as big as when the medium is not magnetisable. If we compare Neumann's formula (25) with equation (33) of § 53, we see that the coefficient of mutual induction between two circuits is really equal to the flux of **B** through circuit 1 when unit current flows in circuit 2. Of course, when $\mu = 1$, **B** and **H** are the same, so that it does not matter whether we speak of the flux of **B** or of **H**; but when $\mu \neq 1$, (25) shows that the flux of **B** is the important quantity which determines the forces exerted by one circuit on another.

In the same way, the coefficient of self-induction L of a single circuit (see § 54) is equal to the flux of **B** through the circuit when unit current is flowing in it.

§ 61. Force on current in a magnetic field

If we replace a large circuit carrying a current by a series of smaller ones, as in § 48, Fig. 33, we see at once from (22) that the potential energy W of an invariable current i in an external magnetic field is

$$W = -i\int \mathbf{B}.d\mathbf{S} = -iN, \quad . \quad . \quad . \ (26)$$

where N * is the total flux of **B** through the circuit. This

* In recent years the flux of **B** is often called ϕ. But as we require ϕ for potential purposes, we shall retain the older symbol N.

formula is fundamental for our later study of fluctuating currents. It also enables us to calculate the force on our coil. For suppose (Fig. 39) that the coil is moved in such a way that an element AB represented by $\mathbf{ds}$ is displaced a distance $\mathbf{t}$ to CD, the current remaining unaltered. Then, from (26) $\delta W = -i$ times the increase in flux of $\mathbf{B}$. We can consider this increase as the sum of the increases in N due to the displacement of all the elements $\mathbf{ds}$ of the original circuit taken separately. But when AB moves to CD the increase in flux of $\mathbf{B}$ is equal to the flux through $ACDB$, which is $\mathbf{B}.\delta\mathbf{S}$, where $\delta\mathbf{S}$ is the area $ACDB$. In order to keep the proper direction for the normal to the surface we must describe the area in the order $ACDB$. Thus $\delta\mathbf{S} = \mathbf{t}\times\mathbf{ds}$. Summing for all the elements $\mathbf{ds}$ we obtain

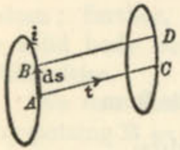


FIG 39.

$$\delta W = \text{increase in potential energy} = -i\int \mathbf{B}.(\mathbf{t}\times\mathbf{ds}).$$

We may re-arrange the triple scalar product * to give

$$\delta W = -i\int \mathbf{t}.(\mathbf{ds}\times\mathbf{B}) \quad . \quad . \quad . \quad (27)$$

Now let us suppose that all elements $\mathbf{ds}$ receive the same displacement $\mathbf{t}$, which may therefore be taken outside the integral in (27). Then, if the total force on the circuit is $\mathbf{F}$,

$$\delta W = -\mathbf{t}.\mathbf{F}.$$

Thus, from (27)

$$\mathbf{F} = i\int \mathbf{ds}\times\mathbf{B} \quad . \quad . \quad . \quad . \quad (28)$$

This shows that the total force on the circuit is the same as if each element $\mathbf{ds}$ experienced a force $i\ \mathbf{ds}\times\mathbf{B}$, which is perpendicular both to $\mathbf{ds}$ and to $\mathbf{B}$. (28) is known as **Ampère's**

* Rutherford, *Vector Methods*, 1946, p. 8.

law, and provides the basis of all electric motors in which we use this force to turn the central rotor shaft. If θ is the angle between $\mathbf{ds}$ and $\mathbf{B}$, Ampère's law is sometimes expressed in the form that the total force is equivalent to a force $iB \sin\theta$ per unit length of wire.

As an example consider the case of two infinite parallel currents i_1 and i_2 a distance a apart in a medium μ. We have seen in § 51 and § 58 (v) that the field $\mathbf{H}$ due to the current i_1 is

$$H_r = 0,\ H_\theta = \frac{2i_1}{r},\ H_z = 0.$$

Thus at all points of the second wire $\mathbf{B}$ has the value $2\mu i_1/a$ perpendicular to the plane of the two wires. Hence, using Ampère's law with $\theta = \pi/2$, the force on the second wire is $2\mu i_1 i_2/a$ per unit length: consideration of its direction shows that this is an attraction if the two currents flow in parallel directions, and a repulsion if in opposite directions.

We have proved (28) for the case of a current flowing in a thin filament. But if we replace $i\ \mathbf{ds}$ by $\mathbf{j}\ dv$, it will immediately apply to the case of a volume distribution of current specified by the current vector $\mathbf{j}$. The mechanical force on such a distribution is therefore

$$\mathbf{F} = \int \mathbf{j}\times\mathbf{B}\, dv \quad . \quad . \quad . \quad (29)$$

This is equivalent to a force $\mathbf{j}\times\mathbf{B}$ per unit volume.

§ 62. Force on a moving charge

Equation (28) shows that the total force on a wire carrying a current i is $i\int \mathbf{ds}\times\mathbf{B}$. This does not prove that the force on each element $\mathbf{ds}$ separately is $i\ \mathbf{ds}\times\mathbf{B}$, since there might be other terms which cancelled on summation. As a matter of fact the force on a tiny element $\mathbf{ds}$ cannot be measured experimentally. But if we assume that it is indeed $i\ \mathbf{ds}\times\mathbf{B}$, we can deduce an expression for the force on a single moving

charge; this is capable of direct experimental verification, and in this way we may justify the splitting up of the total force on a circuit into separate forces on each element $\mathbf{ds}$.

We shall find it best to use (29) rather than (28), so that we assume that the force on unit volume of current really is $\mathbf{j}\times\mathbf{B}$. If we put $\mathbf{j} = Ne\mathbf{v}$,* as in (1) of § 33, we see that the force on a single charge e moving with velocity $\mathbf{v}$ must be

$$\text{force on charge} = e\mathbf{v}\times\mathbf{B} \quad . \quad . \quad . \quad (30)$$

This last formula supposes that e, $\mathbf{v}$, $\mathbf{j}$, $\mathbf{B}$ are measured in e.m.u. If, as is usual, e is in e.s.u. and the rest in e.m.u., the formulæ would have been

$$\text{force on circuit} = \mathbf{F} = \frac{i}{c}\int \mathbf{ds}\times\mathbf{B}, \quad . \quad . \quad (31)$$

$$\text{force on charge} = \frac{e}{c}\,\mathbf{v}\times\mathbf{B}. \quad . \quad . \quad . \quad (32)$$

A moving charge therefore experiences a force proportional to its velocity and perpendicular to $\mathbf{v}$ and $\mathbf{B}$. This formula has been completely verified experimentally in the deflection of electrons in a cathode ray tube, and of heavier charges in a mass spectrograph. It would, in fact, have been possible to start with (32) as an experimental result and from it to build up our earlier theory, though in reverse order.

§ 63. Faraday's disc

An interesting application of (30), which can also be regarded as a further test of its validity, is found in Faraday's disc (Fig. 40). This is a thin conducting disc which rotates in free space with angular velocity p about its axle. There is a uniform magnetic field $\mathbf{H}$ perpendicular to the plane of the disc. A and B are electrical connections to the axle and rim of the disc respectively. On joining A and B by a wire it is found that a current flows, indicating that A and B are

* N is here the number of charges per unit volume, and must not be confused with the flux N in (26) and (33).

at different potentials. The explanation is obtained by considering an electron of charge e at a point P between A and B. If $OP = r$, the velocity of the electron is rp in a direction perpendicular both to the field and to the line OP. The force on this electron is therefore $\pm erpH$ in the direction from A to B, the question of sign being determined by whether the field is into or out of the plane of the disc. Thus the work done in taking a charge e from A to B is

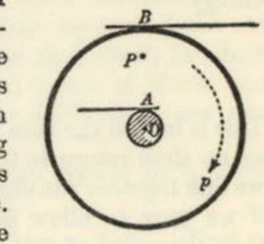


FIG. 40

$$\pm\int_a^b erpH\,dr = \pm\frac{1}{2}\,epH(b^2-a^2),$$

where a and b are the radii of the axle and disc respectively. The potential difference V between A and B is therefore

$$V = \pm\frac{1}{2}\,pH(b^2-a^2) = nN, \quad . \quad . \quad . \quad (33)$$

where n is the number of revolutions per second and N is the total flux of $\mathbf{H}$ across the disc. It is this potential difference that gives rise to the observed current when A and B are connected by a wire. There is, of course, an equal and opposite force on the positively charged nuclei of the disc. But unlike the conduction electrons, these are not free to move and the effect of the force on them is merely a small elastic deformation of the wheel.

This apparatus was used by Lorenz in 1873 to measure the absolute resistance of a wire in e.m.u.

§ 64. Magnetic energy

The close parallel between formulæ of electrostatics and magnetism suggests strongly that corresponding to the

electrostatic energy $\int \frac{\mathbf{D}.\mathbf{E}}{8\pi}\,dv$ there must be a magnetostatic energy

$$\int \frac{\mathbf{B}.\mathbf{H}}{8\pi}\,dv \quad . \quad . \quad . \quad . \ (34)$$

This is indeed the case. We shall not prove the formula here, as we shall return to the matter again in Chapter XIII when we link together the electric and magnetic energies. However, if we were to allow ourselves to introduce magnetic poles, then the proof of (34) would be precisely the same as in the electrostatic case (§ **28**), except that instead of charges we write poles, and instead of electric field and displacement we write magnetic field and induction.

Equation (34) shows us that we may regard the energy as distributed throughout all space with density $\frac{\mathbf{B}.\mathbf{H}}{8\pi}$. Just as in the electrostatic case we could regard this energy as being due to some sort of stress in the æther, and we can show, similarly to § **30**, that a satisfactory stress system is obtained if we suppose that each tube of $\mathbf{B}$ exerts a longitudinal tension $\frac{\mathbf{B}.\mathbf{H}}{8\pi}$ per unit area of cross-section, and a transverse repulsion on adjoining tubes of the same magnitude per unit area of contact. In this way, once again, we are able to dispense with the idea of action-at-a-distance.

There is an important application of (34) in calculating the magnetic energy stored in the medium when a current i flows in a closed coil. As in the electrostatic parallel, we may, if we like, regard this energy as stored up as a result of some form of elastic strain of the æther; but it is wisest not to ask for too particular a model of what is happening, and we shall be content simply to refer to the energy in the medium. It is, however, essential to recognise that the energy is in the medium and not in the magnetic shell; as we have already stressed in Chapter VI, the shell is primarily a mathematical

device, having no kind of real objective existence. Its value is that it allows us to describe the magnetic field $\mathbf{H}$ in an extremely compact manner. Thus, when calculating the energy by integrating (34) over all space, we must leave out that little bit of space occupied by the shell itself, in which indeed on account of the hypothetical layers of North and South poles on the two faces of the shell there are discontinuities in the field.

We show in Fig. 41 a section of the system by a plane which cuts the coil in x, x', and which intersects the magnetic shell in the lightly-shaded area. Since the shell spans the coil, its section in Fig. 41 must be bounded by x and x'.

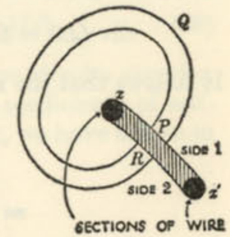


FIG. 41

Consider a tube of induction PQR which leaves the shell at P, but which does not necessarily lie entirely in the plane of xx'. If N is the total flux of $\mathbf{B}$ across the coil, we may write dN for the flux in this tube. The contribution to $\frac{1}{8\pi}\int \mathbf{B}.\mathbf{H}\,dv$ from the volume inside this tube may be found by writing $\mathbf{H} = -\text{grad}\,\Omega$, and $dv = \mathbf{ds}.\mathbf{dS}$, where $\mathbf{ds}$ is an element of length along the tube, and $\mathbf{dS}$ is an element of cross-sectional area. Since the strength of the tube is constant, $\mathbf{B}.\mathbf{dS} = dN$. Thus the contribution to the energy is

$$-\frac{1}{8\pi}\int \text{grad}\,\Omega.\mathbf{ds}\,dN$$
$$= -\frac{dN}{8\pi}\int \frac{\partial\Omega}{\partial s}\,ds$$
$$= -\frac{dN}{8\pi}\,[\Omega],$$

where $[\Omega]$ is the increase in Ω on going from P to R. Now in (8) of Chapter VI we showed that $[\Omega] = -4\pi i$, so the energy in

the tube PQR is $\frac{1}{2}idN$. By addition over all such tubes we see that the total magnetic energy in the medium is

$$\tfrac{1}{2}iN. \quad . \quad . \quad . \quad . \quad . (35)$$

It is possible to deduce (35) without considering each tube separately, as the following analysis shows. We have to evaluate $\frac{1}{8\pi}\int \mathbf{B}.\mathbf{H}\, dv$ through all space outside the shell itself, as in Fig. 41. Let us put $\mathbf{H} = -\text{grad}\ \Omega$, and use the fact that since div $\mathbf{B} = 0$, therefore

$$\text{div}\ \Omega\mathbf{B} = \Omega\ \text{div}\ \mathbf{B} + \mathbf{B}.\text{grad}\ \Omega = \mathbf{B}.\text{grad}\ \Omega.$$

It follows that the magnetic energy is

$$-\frac{1}{8\pi}\int \mathbf{B}.\text{grad}\ \Omega\, dv$$
$$= -\frac{1}{8\pi}\int \text{div}\ \Omega\,\mathbf{B}\, dv$$
$$= -\frac{1}{8\pi}\int \Omega\mathbf{B}.d\mathbf{S}, \quad . \quad . \quad . \quad . (36)$$

where the surface integral is taken outward across the sphere at infinity, and into the shell along the faces at all points such as P and R. A consideration of orders of magnitude shows that at infinity the surface integral vanishes. If we call the two sides of the shell 1 and 2, and let $d\mathbf{S}$ be directed from 2 to 1, (36) reduces to

$$\frac{1}{8\pi}\int\left\{(\Omega\mathbf{B})_1-(\Omega\mathbf{B})_2\right\}.d\mathbf{S}.$$

Now the change in Ω between 1 and 2 is $4\pi i$, and $\mathbf{B}$ is unchanged. So the integral is just

$$\tfrac{1}{2}\, i\int \mathbf{B}.d\mathbf{S} = \tfrac{1}{2}\, iN,$$

as in (35), where N is the total flux of $\mathbf{B}$ across the coil.

It is not difficult to extend this latter argument to the case of several distinct circuits carrying currents i_1, i_2 . . . In fact, if the values of the flux across the circuits are N_1, N_2 . . . the total energy is

$$\tfrac{1}{2}\{N_1i_1+N_2i_2+\ .\ .\ .\} \quad . \quad . \quad . (37)$$

Two particular cases of (37) are important. The first is when there is just one coil, and the flux of $\mathbf{B}$ across it is solely due to the current i in the coil. Then (§ 54), $N = Li$, where L is the coefficient of self-induction, so that

$$\text{energy of a single coil} = \tfrac{1}{2}Li^2 \quad . \quad . (38)$$

The second particular case is that of two circuits carrying currents i_1 and i_2. If L_1 and L_2 are the coefficients of self-induction, and M_{12} is the mutual induction, we have shown in § 54 that

$$N_1 = L_1i_1+M_{12}i_2,$$
$$N_2 = L_2i_2+M_{12}i_1.$$

So the energy due to the two coils together is

$$\tfrac{1}{2}\{N_1i_1+N_2i_2\} = \tfrac{1}{2}\{L_1i_1^2+2M_{12}i_1i_2+L_2i_2^2\}\ . \quad . (39)$$

Comparison of (38) and (39) shows that this total energy is the result of terms for each coil separately and a term $M_{12}i_1i_2$ representing the extra energy due to the mutual interaction of the two currents.

The student will probably wonder why we appear to have two distinct formulæ for the mutual energy of two coils. According to (39) this energy is $+M_{12}i_1i_2$; but according to § 54, equation (32), it is $-M_{12}i_1i_2$. The explanation of the apparent discrepancy is that the two formulæ represent quite different conditions. In deducing (39) we answered the question: "When the currents are i_1 and i_2, how much magnetic energy is stored in the medium?" But in the earlier equation we answered a different question: "If the currents i_1 and i_2 are kept invariable, what potential function may be used to describe the forces exerted by one coil on the

other?" Now when one coil is moved relative to the other, keeping the currents constant, the energy in the medium changes on account of the alteration in M_{12}: this is what is being calculated in (39). But, as we shall see in Chapter XI, electrical work has to be done in order to keep the currents constant during the movement of the one coil. It is this extra work which is responsible for the difference between the two calculations, and may actually be determined from this difference. Indeed, the situation envisaged in the invariable-current formula is rather artificial and practically never occurs. But the situation represented by (38) and (39) is of the greatest importance; when we consider varying currents more fully in Chapters XI and XII we shall find that these two equations are able to provide a simple understanding of the fundamental problem of electromagnetic induction.

§ 65. Examples

1. Show that lines of **B** are refracted at a change of medium, and that if the angles made with the normal are θ_1 and θ_2, then

$$\mu_1 \cot \theta_1 = \mu_2 \cot \theta_2.$$

2. A current i per unit width flows along the boundary between two media of permeabilities μ_1 and μ_2. Show that the boundary conditions are the same as (7) except that the tangential components of **H** in a direction perpendicular to the current satisfy the condition

$$H_{t_1} - H_{t_2} = 4\pi i.$$

3. A torus, or anchor ring, is obtained by rotating a circle about an axis in its own plane distant f from the centre. The volume so formed is filled with soft iron of permeability μ. A total of N turns of wire are wound closely and evenly round the ring, and the wire carries a current i. Show that at points inside the ring the magnetic induction is $2N\mu i/r$, where r is the distance from the axis of rotation. A small air-gap is now made in the iron by cutting away a thin sector bounded by two planes through the central axis which make a small angle α with each

other. Show that the total induction in the air-gap is thus reduced by a factor $\left(1+(\mu-1)\dfrac{\alpha}{2\pi}\right)^{-1}$. This is one way of creating a strong magnetic field by means of a current, if α is kept small.

4. A coil of wire with m turns is closely wound on a circular ring of any form of cross-section with an iron core. A second coil of n turns is intertwined with the first. Show that the coefficient of mutual induction is $M_{12} = 2\mu mn \displaystyle\int \frac{dS}{\rho}$, where ρ is the distance of any point in the cross-section from the axis of the ring. If the cross-section is a circle of radius a, whose centre is R from the axis of the ring, deduce that $M_{12} = 4\pi\mu mn(R-\sqrt{(R^2-a^2)})$.

5. Show that the mutual induction between a circle of radius a and an infinite straight line in the same plane is

$$4\int_{-a}^{+a} \frac{\sqrt{(a^2-x^2)}}{f+x}\, dx,$$

where f is the shortest distance from the centre of the circle to the straight line. Show that integration gives $M_{12} = 4\pi\{f-\sqrt{(f^2-a^2)}\}$.

6. Prove that the force on a magnetic particle of moment **M** in an inhomogeneous field is $(\mathbf{M}.\mathrm{grad})\mathbf{B}$. (Compare Chapter III, question 13.)

7. Use the result of the last question to show that the force **F** on a coil carrying a current i is $\mathbf{F} = i\displaystyle\int (d\mathbf{S}.\mathrm{grad})\mathbf{B}$. Show that if **t** is an arbitrary constant vector this may be written

$$\mathbf{F}.\mathbf{t} = i\int d\mathbf{S}.\mathrm{grad}\,(\mathbf{B}.\mathbf{t}).$$

Next prove that grad $(\mathbf{B}.\mathbf{t}) = $ curl $(\mathbf{B}\times\mathbf{t})$, and hence show that, if μ is constant,

$$\mathbf{F} = i\int d\mathbf{s}\times\mathbf{B}.$$

This is another proof of Ampère's law (28).

8. A current i flows in a circular loop of radius a and a current

i' in an infinitely long straight wire in the same plane. Show that the force between the two currents is

$$4\pi i i' \left\{ \frac{f}{\sqrt{(f^2-a^2)}} - 1 \right\},$$

where f is the shortest distance from the centre of the circle to the wire.

9. Calculate the force on a current i in a coil by the following method. Let AB be a very small element of a coil $ABCA$ which is displaced a distance $\mathbf{t}$ to $A'B'C'A'$. Then using the formula (23) for the Potential Energy, viz. $W = -i \int \mathbf{A}.d\mathbf{s}$, show that $\delta W = i \int \mathbf{A}.d\mathbf{s}$ round the contour $BCAA'C'B'B$. Transform this by Stokes' theorem and hence deduce Ampère's law (28).

10. Currents i and i' flow in two circular loops of area S and S' which are parallel and coaxial a distance c apart. If the one loop is small, show that the attraction between the loops is $6ii'SS'c/R^5$, where R is the distance from the centre of the small loop to the circumference of the larger loop.

11. Use equations (18) and (19) to show that very close to a wire carrying a current i the lines of B and H are circular and deduce that at small distances r from the wire $H = 2i/r$. Use this result to show that two similar loops of wire carrying currents in the same direction, placed nearly in coincidence, attract one another. Notice that this is the reverse of the electrical case where like charges repel; it indicates a fundamental difference between the energy of the fields due to charges and currents.

12. Show that the force exerted on a wire carrying a current i_1 due to a current i_2 in a second wire, may be expressed in the form

$$\mu i_1 i_2 \iint \frac{d\mathbf{s}_1 \times (d\mathbf{s}_2 \times \mathbf{r})}{r^3},$$

where $d\mathbf{s}_1$ and $d\mathbf{s}_2$ are vector elements of the two wires and $\mathbf{r}$ is the vector from $d\mathbf{s}_2$ to $d\mathbf{s}_1$. Apply this formula to calculate the force between two parallel wires.

13. Show that the formula of the last equation may be written

$$\mu i_1 i_2 \iint (d\mathbf{s}_1 . d\mathbf{s}_2) \operatorname{grad} \frac{1}{r} - \mu i_1 i_2 \iint \left(d\mathbf{s}_1 . \operatorname{grad} \frac{1}{r} \right) d\mathbf{s}_2.$$

Verify by integrating first with respect to s_1 that the second term is zero, so that the force is

$$\mu i_1 i_2 \iint (d\mathbf{s}_1 . d\mathbf{s}_2) \operatorname{grad} \frac{1}{r}.$$

This means that each pair of elements ds_1 and ds_2 at an angle θ may be regarded as contributing a force $\mu i_1 i_2 \cos\theta\, ds_1 ds_2/r^2$.

14. An electron of mass m and charge e is projected in any direction into a uniform magnetic field H. If the field H is taken to be parallel to the z axis, show that the equations of motion of the electron are $m\ddot{x} = eH\dot{y}$, $m\ddot{y} = -eH\dot{x}$, $m\ddot{z} = 0$. Deduce that in general the path of the electron is a helix described at constant speed.

15. An electron of mass m and charge e is projected with velocity v in a direction perpendicular to a uniform magnetic field $\mathbf{H}$. Show that it describes a circle of radius ρ, where $H\rho = mv/e$. This is an important result in studies of atomic disintegration by the Wilson Cloud Chamber photographs.

16. Show that the magnetic fields inside the three cavities of § 59 may be calculated by introducing Poisson's Imaginary Magnetic Matter (§ 57) at the boundaries of the cavities, to replace the magnetisation $\mathbf{I}$.

17. Show that the boundary conditions (20) and (21) for $\mathbf{A}$ lead to the more familiar conditions (7) for $\mathbf{B}$ and $\mathbf{H}$.

18. An electron of charge e is moving in free space with constant velocity $\mathbf{v}$. It may be regarded as a small conducting sphere of radius a. Use the value of the associated magnetic field $\mathbf{H}$ given in § 55, question (16), viz. $\mathbf{H} = e\dfrac{\mathbf{v}\times\mathbf{r}}{r^3}$, to show that the total magnetic energy associated with it is $e^2v^2/3a$. Deduce that if the "mass" m of the electron is supposed to be entirely electromagnetic, then $m = 2e^2/3a$. If $e = 1{\cdot}60\times10^{-20}$ e.m.u., and $m = 9{\cdot}11\times10^{-28}$ gms., show that the radius of the electron is about 2×10^{-13} cms.

CHAPTER VIII

PERMANENT MAGNETISM

§ 66. Small magnets

In some substances, such as iron, there is a magnetic moment, measured by its intensity of magnetisation, even when no current is flowing to induce magnetism. In such a case we say that the substance is a **permanent magnet**. In this chapter we shall discuss such magnets. We shall make the further assumption that they are ideally hard; that is, their magnetism cannot be affected by any applied magnetic field. This assumption, which is equivalent to saying that the magnetic susceptibility κ is zero, is not completely correct, and it will prevent us from discussing certain phenomena such as magnetic hysteresis. But it will enable us to understand and to discuss quantitatively many of the properties of permanent magnets.

The smallest unit of permanent magnetism must be the individual atoms in which tiny currents give rise to magnetic properties (§ 3). It is found, however, that in the substances which show permanent magnetism, a large number of such atoms are linked together with their tiny currents similarly oriented, so that the smallest effective magnetic unit consists of a block or **domain** of atoms * which behaves as a single particle. One explanation of this phenomenon is that under certain circumstances the fields of the individual atoms of a block combine together, thus strengthening or co-operating with one another in aligning them all in a certain direction. This is the hypothesis of the **internal field**, due to Weiss.

* Perhaps 1000, but usually more than 10^{10}; yet even then the block is still small on a macroscopic scale, being about 10^{-13} c.cm. in volume.

Within each block there is complete, or almost complete, spontaneous magnetisation. In the presence of a suitable external magnetic field the domain rotates as a whole, and not as a series of separate atoms, thus maintaining its saturation magnetism.

We can describe such a particle by its magnetic moment $\mathbf{m}$. It is irrelevant whether we think of this as due to the individual atomic currents, or suppose that there are two magnetic poles at either end of the particle, giving rise to a magnetic dipole moment of magnitude $\mathbf{m}$. We shall find both points of view useful, though the former is the more accurate.

The effects of a single magnetic particle $\mathbf{m}$ are completely specified by the magnetostatic potential Ω_P at P, to which it gives rise. Just as in the case of an electric dipole (§ 18) we may write any of the following forms :—

$$\Omega_P = \frac{m\cos\theta}{r^2} = \frac{\mathbf{m}.\mathbf{r}}{r^3} = -\mathbf{m}.\mathrm{grad}_P\frac{1}{r} = +\mathbf{m}.\mathrm{grad}_Q\frac{1}{r}, \quad (1)$$

where $\mathbf{r}$ is measured from the dipole Q to the point P (Fig. 42).

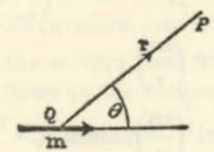


Fig. 42

The corresponding magnetic field $\mathbf{H}$ is obtained from the usual law

$$\mathbf{H} = -\mathrm{grad}\,\Omega \quad . \quad . \quad (2)$$

The component H_r along the radius vector is

$$H_r = -\frac{\partial\Omega}{\partial r} = \frac{2m\cos\theta}{r^3}, \quad . \quad . \quad . \quad (3)$$

and the component H_θ perpendicular to the radius vector is

$$H_\theta = -\frac{1}{r}\frac{\partial\Omega}{\partial\theta} = \frac{m\sin\theta}{r^3}. \quad . \quad . \quad . \quad (4)$$

There does not seem to be any obvious way of calculating the magnetic vector potential $\mathbf{A}$, defined in this case by

$$\operatorname{curl}\mathbf{A} = \mathbf{H}, \quad \operatorname{div}\mathbf{A} = 0. \quad . \quad . \quad . \quad (5)$$

Probably the most direct method is to replace the small magnet by an equivalent current i in a small circular coil of radius a, where $\pi a^2 i = m$. $\mathbf{A}$ may then be calculated by the standard formula $\mathbf{A} = i\int\frac{d\mathbf{s}}{r}$. This has already been done in § 55, questions (11) and (12), with the result

$$\mathbf{A} = \frac{\mathbf{m}\times\mathbf{r}}{r^3}, \quad . \quad . \quad . \quad . \quad (6)$$

$\mathbf{r}$ being measured from the dipole to the point P. Alternatively, with the notation of fig. 42

$$\mathbf{H}_P = -\operatorname{grad}\frac{\mathbf{m}.\mathbf{r}}{r^3} = \operatorname{grad}\left(\mathbf{m}.\operatorname{grad}_P\frac{1}{r}\right)$$

$$= \operatorname{grad}\operatorname{div}\left(\frac{\mathbf{m}}{r}\right)$$

$$= \operatorname{curl}\operatorname{curl}\left(\frac{\mathbf{m}}{r}\right), \text{ since } \nabla^2\frac{\mathbf{m}}{r} = 0$$

$$= \operatorname{curl}\left(\operatorname{grad}\frac{1}{r}\times\mathbf{m}\right)$$

$$= \operatorname{curl}\frac{\mathbf{m}\times\mathbf{r}}{r^3}$$

So we may take $\mathbf{A} = \frac{\mathbf{m}\times\mathbf{r}}{r^3}$, provided that this satisfies $\operatorname{div}\mathbf{A} = 0$. This is soon verified, so that $\mathbf{A}$ is correctly given by (6). The reader may find it easier to remember this formula if it is compared with the corresponding formula for Ω. Thus

$$\Omega = \frac{\mathbf{m}.\mathbf{r}}{r^3}, \quad \mathbf{A} = \frac{\mathbf{m}\times\mathbf{r}}{r^3}.$$

If an external magnetic field $\mathbf{H}$ acts on a magnetic particle $\mathbf{m}$, it will exert a couple. This couple is the same as would be experienced by a small coil carrying the appropriate current, and this may be calculated by using Ampère's rules (§ 44). In fact, since forces on small coils are of the same type as forces on electrical dipoles, which were discussed in § 19, it follows that in a field $\mathbf{H}$ the couple on a small magnet $\mathbf{m}$ is:

$$\text{couple} = \mathbf{m}\times\mathbf{H} \quad . \quad . \quad . \quad (7)$$

Similarly its potential energy in the field is:

$$\text{potential energy} = -\mathbf{m}.\mathbf{H} \quad . \quad . \quad (8)$$

If $\mathbf{H}$ is not constant, there is also a force $\mathbf{F}$ on the magnet. Let us write w for the potential energy (8), so that

$$\mathbf{F} = -\operatorname{grad} w = \operatorname{grad}(\mathbf{m}.\mathbf{H}).$$

If we expand * grad $(\mathbf{m}.\mathbf{H})$ it gives

$$\mathbf{F} = (\mathbf{m}.\operatorname{grad})\mathbf{H} + (\mathbf{H}.\operatorname{grad})\mathbf{m} + \mathbf{m}\times\operatorname{curl}\mathbf{H} + \mathbf{H}\times\operatorname{curl}\mathbf{m}.$$

Since $\mathbf{m}$ is constant the second and fourth terms are zero: and since no current flows at the magnet, $\operatorname{curl}\mathbf{H} = 0$. So

$$\mathbf{F} = (\mathbf{m}.\operatorname{grad})\mathbf{H} \quad . \quad . \quad . \quad (9)$$

The mutual potential energy W between two small magnets $\mathbf{m}$ and $\mathbf{m}'$ separated by a distance $\mathbf{r}$ has been found in (18) of § 19, though the formula there was obtained for electric dipoles. In fact,

$$W = \frac{(\mathbf{m}.\mathbf{m}')}{r^3} - \frac{3(\mathbf{m}.\mathbf{r})(\mathbf{m}'.\mathbf{r})}{r^5} \quad . \quad . \quad (10)$$

From this formula all the forces and couples exerted by either magnet on the other are easily found.

* See Rutherford, *Vector Methods*, 1946, p. 69, or else work in xyz co-ordinates, taking $\mathbf{m}$ along the z axis.

§ 67. A worked example

As an example of the use of these formulæ consider a small magnet **m** placed at P, a distance z (Fig. 43) along the axis of a loop of wire of radius a, carrying a current i. We have seen in (12) of § 49 that the magnetic field at P due to the current is

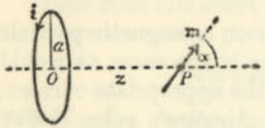


FIG. 43

$$H = 2\pi i a^2/(a^2+z^2)^{\frac{3}{2}}$$

along the axis of the coil. The magnet will therefore experience a couple which, according to (7), is

$$\frac{2\pi i a^2 m}{(a^2+z^2)^{\frac{3}{2}}}\sin\alpha,$$

tending to reduce α, the angle between **m** and the axis of the coil. Let us suppose that the centre of the magnet is fixed at P, but that the magnet is free to rotate about P. Then, if $\mathbf{I}$ is its moment of inertia, the equation of motion is

$$\mathbf{I}\ddot{\alpha} = \text{couple} = -\frac{2\pi i a^2 m}{(a^2+z^2)^{\frac{3}{2}}}\sin\alpha.$$

If α is small, this may be written

$$\ddot{\alpha} = -n^2\alpha, \text{ where } n^2 = 2\pi i a^2 m/\mathbf{I}(a^2+z^2)^{\frac{3}{2}}. \qquad (11)$$

(11) shows that the oscillations are simple harmonic with period $2\pi/n$. If i is known a measurement of the period enables us to calculate m.

Next suppose that the direction of **m** lies along the axis OP. There is then no couple on the magnet, but there is a force F which, using (9), we can write

$$F = m\frac{\partial H}{\partial z} = -\frac{6\pi i a^2 m z}{(a^2+z^2)^{\frac{5}{2}}} \qquad (12)$$

along the axis of the coil.

§ 68. Large magnets

A large permanent magnet is to be regarded as an agglomeration of small magnetic particles. We may describe it in terms of the intensity of magnetisation. We shall call this permanent magnetisation **M** in order to distinguish it from the induced magnetisation **I** which was introduced in § 56 to account for the magnetising effect of an external field. Thus each element of volume dv is equivalent to a magnetic dipole of moment $\mathbf{M}\,dv$. By writing the moment in this form we have effectively "smoothed-out" the large local variations both in magnitude and direction that must actually occur among the individual magnetic units and in the interstices between them. This will not affect the forces exerted outside the magnet, but it does make our discussion of what happens inside a little unreal and arbitrary. This applies particularly to the question of internal energy, as we shall see later.

We may therefore suppose that a permanent magnet is defined by its intensity of magnetisation **M**. This need not necessarily be constant either in magnitude or direction, but we have already decided to consider only cases where **M** is completely unaffected by any external magnetic field. Thus the essential difference between this chapter and the last is that we shall now suppose **M** to be a known function, independent of **H**, whereas in dealing with induced magnetism we supposed that **I** varied and was directly proportional to **H**. With this alteration most of the previous chapter can be adapted to the present case.

Thus, there is still a magnetostatic potential. If we write $\mathbf{M}\,dv$ for **m** in (1) and integrate we see that

$$\Omega_P = \int \mathbf{M}_Q.\text{grad}_Q\frac{1}{r}\,dv_Q \qquad (13)$$

Putting $\text{div}\,\dfrac{\mathbf{M}}{r} = \dfrac{1}{r}\,\text{div}\,\mathbf{M} + \mathbf{M}.\text{grad}\,\dfrac{1}{r}$, and transforming by

the divergence theorem, we obtain

$$\Omega_P = -\int \frac{\operatorname{div} \mathbf{M}}{r}\, dv + \int \frac{\mathbf{M}.d\mathbf{S}}{r} \quad . \quad . \ (14)$$

It appears, as in § 57, that when calculating Ω we may replace the magnetisation by Poisson's imaginary magnetic matter: (14) shows that this is a volume distribution of magnetic poles equal to $-\operatorname{div} \mathbf{M}$ and a surface distribution M_n. The transformation from (13) to (14) is a great help; for if we try to evaluate (13) by direct integration when P is an internal point, there are difficulties of convergence which do not apply to (14).* We can, however, use (13) for outside points without any difficulty.

Thus at points so far away from all the magnets that $1/r$ may be regarded as effectively the same for all the magnetic particles (13) gives

$$\Omega_P = \int \mathbf{M}_Q.\operatorname{grad}_Q \frac{1}{r}\, dv_Q = -\int \mathbf{M}_Q.\operatorname{grad}_P \frac{1}{r}\, dv_Q$$

$$= -\operatorname{grad}_P \frac{1}{r}.\int \mathbf{M}_Q dv_Q = -\operatorname{grad}_P \frac{1}{r}.\ \mathsf{M},$$

where $\mathsf{M} = \int \mathbf{M}\, dv$ is the total magnetic moment. The system therefore behaves at large distances like a single magnetic particle of moment M.

Reasoning exactly as in § 57 we are led to introduce the magnetic induction $\mathbf{B}$, which satisfies the equations

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}, \quad . \quad . \quad . \ (15)$$

$$\operatorname{div} \mathbf{B} = 0, \quad . \quad . \quad . \ (16)$$

B_n and H_s continuous at a change of medium . . (17)

If the medium had not been ideally hard, but had a

* This matter is quite involved, and for a complete account of the difficulties, consult Leathem, *Volume and Surface Integrals used in Physics*, Cambridge Mathematical Tracts No. 1.

permeability μ differing from unity (15) would have been written

$$\mathbf{B} = \mu\mathbf{H} + 4\pi\mathbf{M} = \mathbf{H} + 4\pi\mathbf{I} + 4\pi\mathbf{M}. \quad . \quad . \ (18)$$

However, we have decided to simplify our discussion in this chapter by always putting $\mu = 1$. At points unoccupied by magnetic matter $\mathbf{B} = \mathbf{H}$.

At places where there is no current the magnetostatic potential exists and satisfies

$$\mathbf{H} = -\operatorname{grad} \Omega. \quad . \quad . \quad . \ (19)$$

Combining (15), (16) and (19), or else directly from (14)

$$\nabla^2\Omega = 4\pi \operatorname{div} \mathbf{M}. \quad . \quad . \quad . \ (20)$$

In particular, at all points not occupied by magnetic matter, i.e. outside the magnets, and at other points also if $\mathbf{M}$ is constant

$$\nabla^2\Omega = 0. \quad . \quad . \quad . \ (21)$$

The magnetic vector potential $\mathbf{A}$ is still defined by

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \quad \operatorname{div} \mathbf{A} = 0, \quad . \quad . \quad . \ (22)$$

from which, together with (15), (16) and (19) it is easy to show that

$$\nabla^2\mathbf{A} = -4\pi \operatorname{curl} \mathbf{M}. \quad . \quad . \quad . \ (23)$$

Equations (15) to (23) may be called the equations of the magnetostatic field.

In order to obtain the vector potential for a large magnet we must integrate the formula (6) for a single magnetic particle. If $\mathbf{r}$ is measured from the point P to the particle (this is a different convention from that used in (6)),

$$\mathbf{A}_P = \int \mathbf{M}_Q \times \operatorname{grad}_Q \frac{1}{r}\, dv_Q. \quad . \quad . \quad . \ (24)$$

Now

$$\operatorname{curl}\left(\frac{\mathbf{M}}{r}\right) = \frac{1}{r} \operatorname{curl} \mathbf{M} + \operatorname{grad} \frac{1}{r} \times \mathbf{M}, \text{ so that}$$

$$\mathbf{A} = \int \frac{\operatorname{curl} \mathbf{M}}{r}\, dv - \int \operatorname{curl}\left(\frac{\mathbf{M}}{r}\right) dv.$$

We now make use of the fact that for any vector **a**,

$$\int \text{curl } \mathbf{a} \, dv = -\int \mathbf{a} \times d\mathbf{S} \quad . \quad . \quad . \quad (25)$$

This may be proved by taking each component separately, or alternatively, if **t** is any constant vector we can write

$$\begin{aligned}\mathbf{t} . \int \mathbf{a} \times d\mathbf{S} &= \int \mathbf{t} . (\mathbf{a} \times d\mathbf{S}) \\ &= \int (\mathbf{t} \times \mathbf{a}) . d\mathbf{S} \\ &= \int \text{div } (\mathbf{t} \times \mathbf{a}) \, dv \\ &= \int \{\mathbf{a} . \text{curl } \mathbf{t} - \mathbf{t} . \text{curl } \mathbf{a}\} dv \\ &= -\mathbf{t} . \int \text{curl } \mathbf{a} \, dv.\end{aligned}$$

But **t** is an arbitrary vector, and so

$$\int \mathbf{a} \times d\mathbf{S} = -\int \text{curl } \mathbf{a} \, dv.$$

This shows that

$$\mathbf{A} = \int \frac{\text{curl } \mathbf{M}}{r} dv + \int \frac{\mathbf{M} \times d\mathbf{S}}{r} \quad . \quad . \quad . \quad (26)$$

It is interesting to compare this last formula with the formula (14) for Ω:

$$\Omega = -\int \frac{\text{div } \mathbf{M}}{r} dv + \int \frac{\mathbf{M} . d\mathbf{S}}{r} \quad . \quad . \quad (14)$$

If there is a finite current distribution **j** as well as permanent magnetism **M**, then by combining (26) and the corresponding result (26) of § 52 for currents, we obtain the one equation

$$\mathbf{A} = \int \frac{\mathbf{j} + \text{curl } \mathbf{M}}{r} dv + \int \frac{\mathbf{M} \times d\mathbf{S}}{r} \quad . \quad . \quad (27)$$

§ 69. A uniform bar magnet

Let us illustrate the results of the last paragraph by considering a bar, of constant circular cross-section; the ends are perpendicular to the length of the rod, a distance l apart, and we suppose that the intensity of magnetisation is constant and equal to M, directed along the bar. This is a very fair model for a common bar magnet.

The potential due to this bar magnet may be obtained from (14). Since div $\mathbf{M} = 0$ the field is the same as if we had a layer of positive poles M at one end and an equal layer of negative poles at the other. At large distances, therefore, the magnet behaves like a single magnetic dipole of moment $\pi a^2 l M$, a being the radius of either end. For closer distances it behaves as if poles $\pm\pi a^2 M$ existed at the two ends. Very close to either pole we must take account of the distribution over the whole circular end. (For the detailed result see questions (18) and (19) at the end of the chapter.)

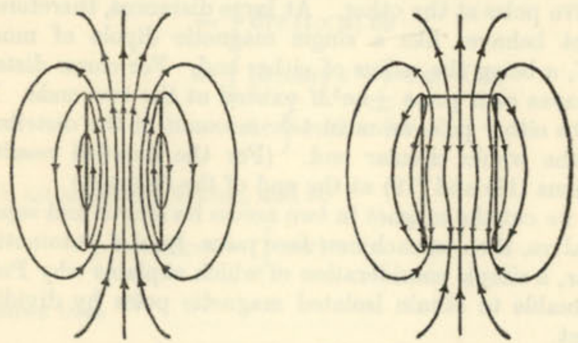
If we cut the magnet in two across its length and separate the halves, then on each new face poles $\pm\pi a^2 M$ automatically appear, a simple consideration of which explains why Poisson was unable to obtain isolated magnetic poles by dividing a magnet.

If we have two long magnets of this kind and by means of corks we make them float horizontally on water in such a way that two of their ends are fairly close and the other two very distant, the interaction of the distant poles may be neglected and we are able, approximately, to measure the forces between isolated magnetic poles. In this way it has been shown that the law of force is the inverse square, within experimental error.

We show in Fig. 44 diagrams illustrating the lines of **B** and **H** for the bar magnet under consideration. The most important fact about these diagrams is that outside the magnet **B** and **H** are identical; but inside since $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$, and B_n is continuous across the ends, **B** and **H** are in opposite directions. The magnetic field **H** inside the magnet is in a

direction opposing the magnetisation **M**, and for this reason it is sometimes referred to as the **demagnetising field** in the magnet. The student should convince himself of the approximate accuracy of these diagrams.

Both diagrams are completely equivalent descriptions. But it is quite clear that **B** rather than **H** is a more natural vector to choose for representing the field. We can understand this if we recall that the atomic magnets from which the magnetisation **M** is compounded are really tiny currents flowing in planes perpendicular to the length of the magnet.



Lines of **B** Lines of **H**

FIG. 44

When we superpose all the tiny currents in the whole bar, the result will be similar to that of § 48, except that instead of splitting up one large circuit into a number of smaller ones (Fig. 33), we are now combining the small currents together. The induction **B** of the bar magnet is therefore the same as if the permanent magnetism had been replaced by a current of magnitude lM flowing round the curved sides of the magnet. It will be recognised at once how closely the lines of **B** resemble the corresponding lines for a long solenoid. Indeed, mathematically, the two calculations are indistinguishable. (See § 72, question 26.)

§ 70. Energy

Since the potential energy of a single magnetic particle in a field **H** is $-\mathbf{m}.\mathbf{H}$, it follows that the potential energy of an invariable magnet is

$$\text{potential energy} = -\int \mathbf{M}.\mathbf{H}\, dv \quad . \quad . \quad (28)$$

In particular if the field **H** is due to another permanent magnet, described by $\mathbf{H}_2$, and the first magnet is described by $\mathbf{H}_1$ and $\mathbf{M}_1$, this energy is

$$-\int \mathbf{M}_1.\mathbf{H}_2\, dv \quad . \quad . \quad . \quad . \quad (29)$$

Since $\mathbf{M}_1 = 0$ except in the space occupied by the first magnet, we may take the integration over all space. We can bring (29) into a more symmetrical form by using the fact that $\int \mathbf{B}_1.\mathbf{H}_2\, dv = 0$ integrated over all space, including the magnets. In fact

$$\int \mathbf{B}_1.\mathbf{H}_2\, dv = -\int \mathbf{B}_1.\text{grad}\, \Omega_2\, dv$$

$$= -\int \{\text{div}\,(\Omega_2\mathbf{B}_1) - \Omega_2\, \text{div}\, \mathbf{B}_1\}\, dv.$$

The last term on the right is zero, since by (16) div $\mathbf{B}_1 = 0$. The other term is transformed by Green's theorem into a surface integral; as there are no discontinuities in $\Omega_2\mathbf{B}_1$ this surface integral is taken over the sphere at infinity only. For a finite magnet considerations of orders of magnitude show that this term is zero. Thus $\int \mathbf{B}_1.\mathbf{H}_2\, dv = 0$, and so (29) may be written

$$\text{potential energy} = -\frac{1}{4\pi}\int (\mathbf{B}_1 - \mathbf{H}_1).\mathbf{H}_2\, dv$$

$$= \frac{1}{4\pi}\int \mathbf{H}_1.\mathbf{H}_2\, dv, \quad . \quad . \quad . \quad . \quad (30)$$

the integration being over all space. The symmetrical form of this result shows that it is also equal to $-\int \mathbf{M}_2.\mathbf{H}_1\, dv$. It is possible, by using (30), to deduce all the forces and couples on either magnet due to the other, assuming always that the magnets are ideally hard, i.e. that the magnetisation $\mathbf{M}_1$ is quite unchanged by the presence of $\mathbf{H}_2$. As we stated earlier, this is only approximately correct.

We can also use (30) to calculate the field energy of a large magnet. For suppose that the large magnet is built up from a series of n separate small magnets. The total field $\mathbf{H}$ is the sum of the partial fields due to each small magnet, so that

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 + \ldots + \mathbf{H}_n \qquad . \quad . \ (31)$$

Now the energy of the field due to the large magnet may be regarded as the work done in bringing these n small magnets together under their own mutual influence ; so it is

$$\frac{1}{4\pi}\int \{\mathbf{H}_1.\mathbf{H}_2 + \mathbf{H}_1.\mathbf{H}_3 + \mathbf{H}_2.\mathbf{H}_3 + \ldots\}\, dv,$$

in which the contribution from each pair of small magnets is included. By (31) this may be written

$$\text{energy} = \frac{1}{8\pi}\int \mathbf{H}^2\, dv - \left\{\frac{1}{8\pi}\int \mathbf{H}_1{}^2 dv + \frac{1}{8\pi}\int \mathbf{H}_2{}^2 dv + \ldots\right\}$$
$$= \int \frac{\mathbf{H}^2}{8\pi}\, dv - C, \quad . \quad . \quad . \quad . \quad . \ (32)$$

where C is a constant independent of the way in which the small units are fitted together: it is therefore a measure of their internal energy when separate from each other. (32) shows that apart from some unknown constant term we may take the field energy of the large magnet to be

$$\text{field energy} = \frac{1}{8\pi}\int \mathbf{H}^2 dv, \quad . \quad . \quad . \ (33)$$

where the integration is over all space, including the magnet. We may speak of an energy density $\mathbf{H}^2/8\pi$ in a manner

similar to that which we have used for electrostatic energy in § 28. Indeed, (33) is precisely the same as would be expected for the energy of the magnetic field according to the discussion in § 64. More accurately, it is the same as would be anticipated in those parts of space outside the actual magnet itself: inside the magnet we should expect to find the product of $\mathbf{B}$ and $\mathbf{H}$ instead of $\mathbf{H}^2$.

The result in (33) is of a familiar form, but we must remember its limitations; besides our assumption that the magnetisation is invariable for each small particle, we have also assumed, in (31), that the principle of superposition (§ 7) is valid, and the whole phenomenon of hysteresis shows that this is not so; nor have we shown how the self-energy of the small particles—C in (32)—depends upon what assumptions we may make with regard to their shape, size and number. For these reasons, (33) is to be regarded as the energy of a theoretical magnet, rather than a physical one. Further discussion of the internal energy of a real magnet is beyond the scope of this book.

§ 71. Terrestrial magnetism

There is one other aspect of permanent magnetism which is of special importance. This is the earth's magnetic field. We are all aware of the use to which it is put in the mariner's compass. A detailed description of the earth's magnetism is very complicated, especially as there are several seasonal variations of different periods, but for many purposes it is good enough to suppose that the earth behaves like a uniformly magnetised sphere, though the direction of magnetisation (the magnetic polar axis, which does not quite coincide with the earth's axis of rotation) is slowly changing. Let us therefore consider the magnetic field of a sphere of radius a (Fig. 45) uniformly magnetised with intensity M in the z direction.

The best way of dealing with this problem is to solve the differential equation for the magnetostatic potential Ω, as

illustrated in the next chapter. This may be done by combining the analysis of §§ 76 and 79. But Ω may also be obtained by direct integration, using Poisson's imaginary magnetic matter (§ 68). According to (14), since div $\mathbf{M} = 0$, there is no volume distribution of poles and the surface distribution is M_n, i.e. $M \cos\theta$ per unit area. Thus

$$\Omega_P = \int \frac{M \cos\theta \, dS}{PQ} \quad . \quad . \quad . \ (34)$$

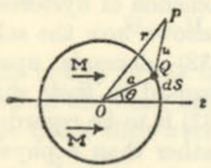


Fig. 45

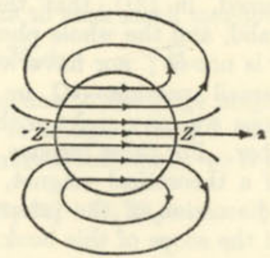


Fig. 46

The proper way of dealing with this integral is to expand $1/PQ$ in a series of Legendre's Polynomials, with OP as polar axis (cf. § 78, equation (12)). On integration all the terms vanish except one and the result (35) is soon obtained. But we give an alternative proof below for the student who is not familiar with these important polynomials.

Let $POz = \alpha$, $POQ = \Theta$, and let us define the position of dS at Q by the variables u and ψ, where $u = PQ$ and ψ is the angle between the planes POz and POQ. It is soon verified that

$$u^2 = a^2 + r^2 - 2ar \cos\Theta,$$
$$\cos\theta = \cos\alpha \cos\Theta + \sin\alpha \sin\Theta \cos\psi,$$
$$dS = a^2 \sin\Theta \, d\Theta \, d\psi = \frac{a}{r} u \, du \, d\psi.$$

We may therefore write (34) in the form

$$\Omega_P = \frac{Ma}{r} \int \{\cos\alpha \cos\Theta + \sin\alpha \sin\Theta \cos\psi\} du \, d\psi$$
$$= \frac{2\pi Ma}{r} \int \cos\alpha \cos\Theta \, du$$
$$= \frac{2\pi Ma \cos\alpha}{r} \int \frac{a^2 + r^2 - u^2}{2ar} du.$$

If P is outside the sphere, the limits of u are $r-a$ and $r+a$: but if P is inside they are $a-r$ and $a+r$. The integration is quite straightforward and the result is:

$$\Omega_P = \tfrac{4}{3} \pi a^3 M \frac{\cos\alpha}{r^2}, \ P \text{ outside the sphere} \quad . \ (35)$$
$$= \tfrac{4}{3} \pi M r \cos\alpha, \ P \text{ inside the sphere} \quad . \quad . \ (36)$$

(35) shows that at outside points the sphere behaves just as if a large magnetic dipole of moment $\frac{4}{3}\pi a^3 M$ was concentrated at the centre, but at inside points (36) shows that the field is uniform, and of strength $\frac{4}{3}\pi M$. The field lines of $\mathbf{B}$ are shown in Fig. 46.

From (35) and (36) the direction and magnitude of the earth's magnetic field are easily obtained. We notice from Fig. 46 that at points on the earth's surface the direction of the field makes a non-zero angle with the horizontal meridian at the point. This angle is called the **angle of dip.** Evidently the dip has its greatest value of 90° at the magnetic poles Z, Z'.

We shall conclude this chapter with an alternative proof of (35) and (36). We mention it because the method of proof may be generalised to apply to any uniformly magnetised body. (See § 72, questions 21 and 22.) Each element dv of the sphere may be regarded as a magnetic dipole of moment $M \, dv$: this in turn may be regarded as a pair of positive and negative poles $\pm \dfrac{M dv}{l}$, a very small distance l apart. In this way we

may split up the whole magnetism into a uniform distribution of positive poles M/l per unit volume filling one sphere, and an equal distribution of negative poles filling an equal sphere, displaced a very small distance l from each other, as shown in Fig. 47. At points outside the spheres each distribution behaves like a concentrated pole of size $\pm \frac{M}{l} \times \frac{4}{3}\pi a^3$, so that the field is that due to a magnetic dipole in which the pole strength is $4\pi a^3 M/3l$, and the separation of the poles is l. The moment of such a dipole is $\frac{4}{3}\pi a^3 M$, in complete agreement with (35). We leave it as an exercise for the reader to prove (36) for internal points in the same way.

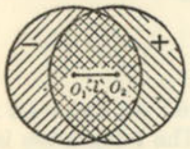


FIG. 47

§ 72. Examples

1. Show that the time of swing of a magnetic dipole of moment m and moment of inertia I in a field H is $2\pi\sqrt{(I/mH)}$.

2. A tangent galvanometer consists of a coil of n turns of wire of radius a, resting with its plane vertical and in the magnetic meridian. The horizontal component of the earth's field is H. Show that when a current i flows round the coil a small compass needle placed at the centre of the coil is deflected through an angle θ where $\tan\theta = 2\pi ni/aH$.

3. A small magnet is placed at distance r from an infinite straight current i. Prove that in addition to a couple, it also experiences a force $2im/r^2$, where m is the projection of the moment upon the shortest distance between the magnet and the current.

4. Deduce the formula $(\mathbf{m}\,.\,\text{grad})\mathbf{H}$ for the force on a magnetic particle in a field $\mathbf{H}$ by regarding the particle as a pair of equal and opposite magnetic poles a small distance apart.

5. Show that if we attempt to calculate the energy of the field due to a single small magnet by integrating $H^2/8\pi$ throughout all space we obtain an infinite value.

6. In order to avoid the difficulty of the previous question, suppose that each elementary magnet is a uniformly magnetised sphere of radius a. Show that $\int \frac{\mathbf{H}^2}{8\pi}\,dv$ is now finite and has a value $\frac{8}{9}\pi^2 a^3 M^2$, where M is the intensity of magnetisation.

7. Two small magnets are free to rotate in a plane about their centres which are fixed. Show that the only stable positions of equilibrium are those in which the magnets are pointing in the same direction along the line joining their centres.

8. Starting from the formula (10) for the mutual potential energy of two small magnets, show that the couple $\mathbf{G}$ exerted on $\mathbf{m}$ is

$$\frac{\mathbf{m}\times\mathbf{m}'}{r^3} - \frac{3(\mathbf{m}\times\mathbf{r})(\mathbf{m}'\,.\,\mathbf{r})}{r^5}.$$

Obtain the same result by writing $\mathbf{G} = \mathbf{m}\times\mathbf{H}$, where $\mathbf{H}$ is the field due to the $\mathbf{m}'$ magnet.

9. In the previous question why is the couple exerted on m by m' not equal and opposite to the couple exerted on m' by m?

10. Verify that the vector $\mathbf{A}$ whose components are

$$\left(\frac{\lambda yz}{(x^2+y^2)r},\ -\frac{\lambda xz}{(x^2+y^2)r},\ 0\right)$$

is such that $\text{curl}\ \mathbf{A} = \left(\frac{\lambda x}{r^3}, \frac{\lambda y}{r^3}, \frac{\lambda z}{r^3}\right)$, and $\text{div}\ \mathbf{A} = 0$. Show that $\mathbf{A}$ would be the vector potential for an isolated magnetic pole of strength λ at the origin. Deduce from this the value of $\mathbf{A}$ for a magnetic dipole at the origin pointing in the z direction. [See (6) in § 66.]

11. Show, by direct vector differentiation of (1) and (6), that $\text{curl}\ \mathbf{A} = \mathbf{H} = -\text{grad}\,\Omega$.

12. Show that if we attempt to calculate the magnetic field-energy of a given permanent magnet by evaluating $\frac{1}{8\pi}\int \mathbf{B}\,.\,\mathbf{H}\,dv$ through all space, the result is zero. Explain this.

13. Show that with two permanent magnets, in the notation of § 70, $\int \mathbf{B}_1\,.\,\mathbf{B}_2\,dv = -\int \mathbf{H}_1\,.\,\mathbf{H}_2\,dv$. Deduce that their mutual potential energy may be written in the form

$$\frac{1}{8\pi}\int \mathbf{H}_1\,.\,\mathbf{H}_2 - \mathbf{B}_1\,.\,\mathbf{B}_2\,dv.$$

14. Show that the potential energy of an invariable magnetic shell of uniform strength p in a given field is $-pN$, where N is the flux of induction of the given field through the shell.

15. Write down the equations to be satisfied by the potential Ω due to a uniformly magnetised sphere, and verify that they are all satisfied by (35) and (36).

16. Prove the result (36) for the potential inside a uniformly magnetised sphere by the method illustrated at the end of § 71.

17. One sphere completely surrounds another. The space between is uniformly magnetised. Show that there is no field inside the cavity. Write down an expression for the magnetostatic potential at any point.

18. Show that at points on the axis of the cylindrical bar magnet discussed in § 69 the induction $\mathbf{B}$ is directed along the axis and has magnitude $2\pi M(\cos\beta \pm \cos\alpha)$, where α and β are the acute angles subtended at the point by radii of the two ends, and the positive sign relates to inside, and the negative sign to outside points. Compare the result with § 49 for the field due to a solenoid.

19. Deduce from the result of question (18) that just outside the centre of the positive face of the bar magnet the field H is

$$2\pi M \frac{l}{\sqrt{(l^2+a^2)}}$$

away from the magnet; but just inside the magnet it is

$$2\pi M\left\{2-\frac{l}{\sqrt{(l^2+a^2)}}\right\}$$

directed from the positive to the negative face. Notice how this confirms the direction of the lines of $\mathbf{H}$ in Fig. 44.

20. If we regard the magnetisation of the bar magnet of questions (18) and (19) to be replaced by two layers of positive and negative poles $\pm M$ per unit area of the two faces, then question (19) gives the magnetic field $\mathbf{H}$ just outside the positive layer $+M$. But if we had been dealing with an electrostatic problem in which a conductor carrying a surface charge σ per unit area of the ends was substituted for the bar magnet, the electric field would be simply $4\pi\sigma$. Explain the difference between these two formulæ.

21. At a point on the axis inside the bar magnet of the last three questions there is a small spherical hole. By using Poisson's

Imaginary Magnetic Matter, or otherwise, show that the field at the centre of this hole is $-\frac{8}{3}\pi M+2\pi M(\cos\beta+\cos\alpha)$. If the magnet is very long deduce that the field is $\frac{4}{3}\pi M$ in a direction opposite to the magnetisation.

22. Show that for a uniformly magnetised solid $\mathbf{A} = \mathbf{M}\times\int\frac{d\mathbf{S}}{r}$.

Next show that for a sphere of radius a and centre O, $\int\frac{d\mathbf{S}}{r}$ is a vector in the direction from O to the point P from which r is measured. If $OP = \mathbf{R}$, prove that

$$\int\frac{d\mathbf{S}}{r} = \frac{\pi\mathbf{R}}{R^3}\int_{|a-R|}^{a+R}(a^2+R^2-r^2)\,dr.$$

Deduce that the magnetic vector potential for a uniformly magnetised sphere is given by

$$\text{Outside the sphere}\quad \mathbf{A} = \frac{4}{3}\pi a^3\frac{\mathbf{M}\times\mathbf{R}}{R^3}$$

$$\text{Inside the sphere}\quad \mathbf{A} = \frac{4}{3}\pi\mathbf{M}\times\mathbf{R}.$$

Interpret your result.

23. The magnetisation $\mathbf{M}$ of a certain body is constant. Show that at a point P,

$$\Omega_P = -\mathbf{M}.\text{grad}_P\int\frac{dv_Q}{r_{PQ}}.$$

If we put V_P for the electrostatic potential at P due to unit volume density of charge throughout the body, show that this may be written

$$\Omega_P = -\mathbf{M}.\text{grad}\,V_P.$$

Deduce equation (35) for the potential of a uniformly magnetised sphere.

24. If the magnetisation $\mathbf{M}$ of a certain body is constant, show that equation (24) may be written

$$\mathbf{A}_P = -\mathbf{M}\times\text{grad}_P\int\frac{dv_Q}{r_{PQ}}.$$

With the notation of the previous question show that this is

$$\mathbf{A} = -\mathbf{M}\times\text{grad}\,V.$$

Deduce the results of question (22) for a uniformly magnetised sphere.

25. Show that the magnetostatic potential Ω for a uniformly magnetised solid is the same, at outside points, as the potential due to a magnetic shell closely surrounding the body and of strength Mz, where z is measured in the same direction as the magnetisation **M**.

26. A cylindrical bar magnet of length l is uniformly magnetised with intensity M. Show that the vector potential given by (26) is precisely the same as we should find for a current lM flowing uniformly round the curved surface of the magnet. (This was stated without proof at the end of § 69.)

27. If the earth be regarded as a uniformly magnetised sphere, and if δ is the dip at a point whose magnetic latitude is α, prove that $\tan \delta = 2 \tan \alpha$.

28. If the earth is a uniformly magnetised sphere of total moment M, prove that an electron of charge e and mass m can describe a circle of radius r outside the magnetic equator if its velocity is $e\mathsf{M}/mr^2$.

CHAPTER IX

POTENTIAL PROBLEMS

§ 73. Mathematical equivalence of all potential problems

The solution of a problem in electrostatics may be regarded as known when we have determined the electrostatic potential at all points. The same is true for the distribution of current, and for magnetism, both permanent and induced. The reader will already have noticed that the differential equations which determine the potential are practically identical in all cases: the differences mainly occur in the boundary conditions. For this reason the mathematics used to solve one problem may often be used immediately to solve another; we have therefore reserved till this chapter a series of problems, closely related, to which we may give the name of potential problems, since they all consist in solving the potential equation under various given conditions. In the next chapter we continue the discussion by outlining some important special methods.

Let us first consider the equations that must be satisfied by a potential function. For convenience we shall call it ϕ, without stating whether the resulting function applies to an electrostatic, magnetostatic or current-flow problem. In practically all cases we have Laplace's equation.

(i) $$\nabla^2\phi = 0.$$

The boundary conditions vary somewhat according to the type of problem. Thus

(ii) $$\phi = \text{constant}$$

is the condition on any conductor in electrostatics or electrode in current flow ; and

(iii) $\int \frac{\partial \phi}{\partial n}\, dS$ is related to the total charge on the conductor or to the total current from an electrode. With a finite system of charges, currents or magnets :

(iv) $\phi \to 0$ at infinity.

But if there is a uniform field E_0 at infinity in the z direction, this latter condition must be replaced by

$$\phi = -E_0 z + \phi', \text{ where } \phi' \text{ is finite at infinity.}$$

This may be put in the form

$$\phi + E_0 z \text{ is finite at infinity.}$$

(v) Further, there can be no singularities in ϕ except at isolated charges, double layers, electrodes or magnets. Near an isolated charge e for example, $\phi = e/Kr + \phi'$, where ϕ' is finite, r being measured from the charge, so that $\phi - e/Kr$ is finite. Similarly at a dipole $\mathbf{m}$ in a vacuum, $\phi - \frac{\mathbf{m}.\mathbf{r}}{r^3}$ is finite.

(vi) At a sudden change of medium other types of boundary, or continuity, conditions are introduced. Thus at the boundary between a non-conductor and a conductor carrying a current,

$$\frac{\partial \phi}{\partial n} = 0.$$

If we have two media, 1 and 2, in contact, we shall generally have two distinct analytical expressions ϕ_1 and ϕ_2 valid in the two regions, with boundary conditions (cf. § 26) at all the common points of 1 and 2,

$$\phi_1 = \phi_2, \quad K_1 \frac{\partial \phi_1}{\partial n} = K_2 \frac{\partial \phi_2}{\partial n},$$

or their equivalent.

§ 74. Methods available

It is quite clear from § 73 that the various types of problem are essentially equivalent. Now let us see what methods are available for finding ϕ. In simple cases, such as an isolated charge, a charged sphere or a small magnet, we know at once the value of ϕ, and the principle of superposition enables us to combine together any number of such solutions. There are also special methods which we discuss in the next chapter. But if our problem does not fall into any of these types, we usually expand ϕ as a series of harmonic functions * with coefficients that must be determined from the boundary conditions.

In spherical polar co-ordinates r, θ, ψ the simplest harmonic function is just $1/r$, in which r may be measured from any arbitrary point. This is sometimes written $\frac{1}{|\mathbf{r}-\mathbf{r}_0|}$, $\mathbf{r}_0$ being an arbitrary constant vector. Other simple harmonic functions are $r \cos \theta$ and $\cos \theta/r^2$. Since $r \cos \theta = z$, these latter are particularly suitable for problems connected with a uniform field, as we should expect from (iv) above. These three functions are particular cases of the spherical harmonics $r^n P_n(\cos \theta)$ and $r^{-(n+1)} P_n(\cos \theta)$, where $P_n(\cos \theta)$ is the Legendre Polynomial of degree n in which n is a positive integer.† Since $P_0(\cos \theta) = 1$, and $P_1(\cos \theta) = \cos \theta$, we recognise at once the nature of our earlier functions. The functions $r^n P_n(\cos \theta)$ remain finite when $r = 0$, but not when $r =$ infinity ; the opposite is true for $r^{-(n+1)} P_n(\cos \theta)$. It is this fact which usually decides which of the two types will be required in a given problem. Both of them have cylindrical symmetry about the polar axis $\theta = 0$. If, however, our problem does not have this degree of symmetry, we require to use tesseral harmonics such as $r^n P_n^m (\cos \theta) \cos m\psi$. However we shall not use such functions in this book.

* Any solution of $\nabla^2 \phi = 0$ is called a harmonic function.

† For a discussion of Legendre's Polynomials, without which this chapter cannot be properly understood, see e.g. Jeans, *Electricity and Magnetism*, or Whittaker and Watson, *Modern Analysis*.

The corresponding functions in two-dimensional polar co-ordinates r, θ are

$$\log r, \quad \log |\mathbf{r}-\mathbf{r}_0|, \quad r^n \cos n\theta, \quad r^n \sin n\theta.$$

With these latter it may be necessary to restrict n to be an integer, positive or negative, if we want to keep ϕ single-valued over the whole range of θ.

In cylindrical polar co-ordinates r, θ, z the standard harmonic functions are $J_m(nr) \cos m\theta\ e^{\pm nz}$, and $J_m(nr) \sin m\theta\ e^{\pm nz}$, where $J_m(nr)$ is the Bessel function of order m and variable nr. We shall not require any of these functions, however, in this chapter, nor any of the modifications which occur in them if m or n is taken to be complex.*

There are, of course, many other harmonic functions besides those enumerated above. But as we shall see below, these are sufficient to deal with a great many important problems. The rest of this chapter will be devoted to illustrations of their application in the various branches of electrostatics, current flow and magnetism.

There is one other point that we must mention here, though we return to it in more detail in the next chapter. If by any means whatever we have found a potential function ϕ which satisfies all the required conditions, this must be the correct solution, for it can be proved (§ 83) that the conditions we have listed in § 73 are sufficient to define ϕ uniquely.

§ 75. Spherical conductor in a uniform field

Consider the case of a conducting sphere of radius a (Fig. 48) placed at zero potential (i.e. connected to earth) in a uniform electric field E_0 parallel to the z axis. If we take spherical polar co-ordinates with the axis along Oz, we have to find a potential ϕ which satisfies the following conditions (see §§ 10, 13 and 73 (iv)) :—

(i) $\nabla^2\phi = 0$,

(ii) $\phi = 0$ when $r = a$, for all values of θ,

(iii) $\phi + E_0 r \cos\theta$ is finite at $r =$ infinity.

* See e.g. Coulson, *Waves*, Oliver and Boyd Ltd., 1949, pp. 14, 76.

The form of (iii) suggests that we try the functions referred to in § 74 which involve $\cos\theta$ times a function of r. We shall see the justification for this later. But for the present let us see if we can get a solution

$$\phi = -E_0 r \cos\theta + \frac{A\cos\theta}{r^2}, \quad \cdot \quad \cdot \quad (1)$$

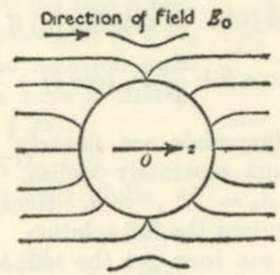


Fig. 48

in which A is an arbitrary constant to be determined. This choice automatically satisfies (i) and (iii). It also satisfies (ii) if

$$-E_0 a + \frac{A}{a^2} = 0, \text{ i.e. } A = E_0 a^3.$$

Thus the solution of this problem is simply

$$\phi = -E_0 r \cos\theta + \frac{E_0 a^3 \cos\theta}{r^2} \quad \cdot \quad \cdot \quad (2)$$

The lines of force corresponding to this potential are shown in the diagram. We may think of the term $E_0 a^3 \cos\theta / r^2$ as representing the disturbing effect of the sphere. This disturbing effect is the same as if a dipole of moment $E_0 a^3$ was placed at 0 pointing in the direction of the field E_0. (Sometimes called the image dipole, see § 86.)

It is obviously true that (2) provides us with a complete solution. But we inevitably ask : why are there no more terms ? To answer this question let us suppose that there

were other terms. As there is cylindrical symmetry about $\theta = 0$, the additional terms must be of the general form $r^{-(n+1)}P_n$ (cos θ). If we included terms $r^n P_n$ (cos θ), then $\phi + E_0 r \cos\theta$ would not be finite at infinity. So let us try

$$\phi = -E_0 r \cos\theta + \frac{A_1}{r^2}\cos\theta + \frac{A_2}{r^3}P_2(\cos\theta) + \dots \quad . \ (3)$$

This satisfies (i) and (iii). It satisfies (ii) if, for all values of θ,

$$0 = \left(-E_0 a + \frac{A_1}{a^2}\right)\cos\theta + \frac{A_2}{a^3}P_2(\cos\theta) + \frac{A_3}{a^4}P_3(\cos\theta) + \dots$$

The Legendre polynomials are linearly independent. So each coefficient must separately vanish. This means that $A_1 = E_0 a^3$, $A_2 = A_3 = \dots = 0$. Hence the extra terms vanish, and (2) is indeed the full solution.

Knowing ϕ we can soon get the induced charge density σ on the surface of the sphere. For

$$4\pi\sigma = -\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \left[E_0\left(1 + \frac{2a^3}{r^3}\right)\cos\theta\right]_{r=a} = 3E_0\cos\theta.$$

Hence

$$\sigma = \frac{3}{4\pi}E_0\cos\theta \quad . \quad . \quad . \quad . \ (4)$$

This shows that the induced charge is negative on the left half of the sphere, and positive on the right half, so that the total charge on the sphere is zero.

If the sphere receives a charge Q it will no longer be at zero potential. But we can determine the resultant potential by the principle of superposition. In fact,

$$\phi = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta + \frac{Q}{r} \quad . \quad . \ (5)$$

The potential of the sphere ($r = a$) is now Q/a, and all other quantities can easily be found. If Q lies between $\pm 3a^2 E_0$ the surface of the sphere will be divided into a region of negative charge and another region of positive charge.

§ 76. Dielectric sphere in a uniform field

Let us replace the conducting sphere of § 75 by a dielectric sphere of uniform dielectric constant K_2 (Fig. 49) placed in a medium of dielectric constant K_1.

There are now two separate regions of space, inside and outside the sphere. Let us call the two potentials ϕ_1 and ϕ_2. According to § 73, and using the same notation as in the last section, we have to find ϕ_1 and ϕ_2 so that (see § 26)

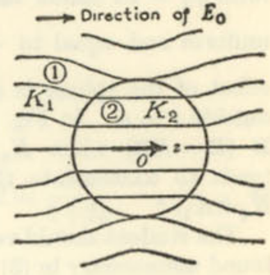


Fig. 49

(i) $\nabla^2\phi_1 = 0$, $\nabla^2\phi_2 = 0$,
(ii) $\phi_1 + E_0 r\cos\theta$ is finite at infinity,
(iii) ϕ_2 finite in $r \leqslant a$,
(iv) $\phi_1 = \phi_2$ on $r = a$, for all θ,
(v) $K_1\dfrac{\partial\phi_1}{\partial r} = K_2\dfrac{\partial\phi_2}{\partial r}$ on $r = a$, for all θ. This is the condition (17) of § 26 that D_n, or KE_n, must be continuous at the change of medium. The results for the conducting sphere suggest that we try

$$\phi_1 = -E_0 r\cos\theta + \frac{A}{r^2}\cos\theta \quad . \quad . \quad . \ (6)$$

$$\phi_2 = Br\cos\theta \quad . \quad . \quad . \quad . \quad . \ (7)$$

This satisfies (i), (ii) and (iii) whatever the values of A and B. It also satisfies (iv) if $-E_0 a + \dfrac{A}{a^2} = Ba$, and (v) if

$$K_1\left\{-E_0 - \frac{2A}{a^3}\right\} = K_2 B.$$

These are two simultaneous equations for A and B, giving

$$A = \frac{K_2 - K_1}{K_2 + 2K_1}E_0 a^3, \quad B = \frac{-3K_1}{K_2 + 2K_1}E_0 \quad . \ (8)$$

(6), (7) and (8) solve our problem completely. Taking $K_1 = 1$

we have the case of a dielectric sphere in a vacuum. Taking $K_2 = 1$ we have the case of a spherical hole in a dielectric K_1. This is interesting as it adds something more to the discussion of cavities in § 31. The fact that ϕ_2 may be written in the form $\phi_2 = Bz$ shows that the field inside the sphere is quite uniform and equal to $-B$, i.e. $\dfrac{3K_1}{K_2+2K_1}E_0$. The disturbing effect of the sphere is shown by the dipole term $A\cos\theta/r^2$ outside $r = a$. In Fig. 49 we show the lines of displacement $\mathbf{D}$ ($\mathbf{D}= K\mathbf{E}$) when $K_2>K_1$. This shows that the sphere tends to concentrate the lines. Just the opposite holds if $K_2<K_1$.

The student should verify that extra terms such as those we found unnecessary in (3) are not needed in this problem either. If we had required them, they would have been of the form

$$\phi_1 = -E_0 r\cos\theta+\frac{A_1}{r^2}\cos\theta+\frac{A_2}{r^3}P_2(\cos\theta)+\frac{A_3}{r^4}P_3(\cos\theta)+\ldots$$

$$\phi_2 = B_1 r\cos\theta+B_2 r^2 P_2(\cos\theta)+B_3 r^3 P_3(\cos\theta)+\ \ldots \qquad (9)$$

§ 77. Small magnet in a spherical hole

Our next example is taken from magnetism. Suppose (Fig. 50) that a small magnet of moment m lies at O pointing along the z axis, at the centre of a spherical hollow of radius a surrounded by a medium of uniform permeability μ. Measuring from O as origin we have to find two potential functions ϕ_1 and ϕ_2, so that (see §§ 66, 68)

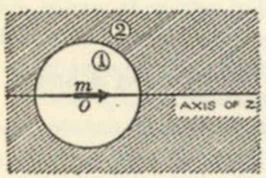


FIG. 50

(i) $\nabla^2\phi_1 = 0$, $\nabla^2\phi_2 = 0$,

(ii) $\phi_2 \to 0$ at infinity,

(iii) $\phi_1 - m\cos\theta/r^2$ is finite at $r = 0$,

(iv) $\phi_1 = \phi_2$ at $r = a$, for all θ,

(v) $\dfrac{\partial\phi_1}{\partial r} = \mu\dfrac{\partial\phi_2}{\partial r}$ at $r = a$, for all θ. This is the condition (17) of § 68, that B_n is continuous at the change of medium.

In view of (iii) it is natural to look for solutions involving $r\cos\theta$ and $\cos\theta/r^2$. We leave it as an exercise to the reader to verify that if we write

$$\phi_1 = \frac{m\cos\theta}{r^2}+A\,r\cos\theta,$$

$$\phi_2 = \frac{B\cos\theta}{r^2},$$

our conditions are all satisfied by

$$A = \frac{2(1-\mu)}{1+2\mu}\frac{m}{a^3},\quad B = \frac{3m}{1+2\mu} \qquad (10)$$

§ 78. Point charge outside a dielectric sphere

Our next problem is a little more difficult. We are to find the potential function due to a point charge e placed at B (Fig. 51) a distance b from the centre of an uncharged dielectric sphere of radius a. The notation is shown in the figure. We have to find two potential functions ϕ_1 and ϕ_2 so that (see § 26)

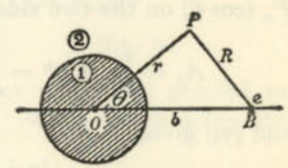


FIG. 51

(i) $\nabla^2\phi_1 = 0$, $\nabla^2\phi_2 = 0$,

(ii) $\phi_2 - \dfrac{e}{R}$ is finite at $R = 0$, R being measured from B,

(iii) $\phi_2 \to 0$ at infinity. More accurately, since the total charge is e, we shall have $\phi_2 = \dfrac{e}{r}$ + higher powers of $\dfrac{1}{r}$, at infinity,

(iv) ϕ_1 is finite if $r \leqslant a$,

(v) $\phi_1 = \phi_2$ on $r = a$, for all θ,

(vi) $K\dfrac{\partial\phi_1}{\partial r} = \dfrac{\partial\phi_2}{\partial r}$ on $r = a$, for all θ. This is the continuity condition for D_n at the change of medium.

This problem is one in which we shall need all the Legendre's functions of § 74. So let us write

$$\phi_1 = A_0 + A_1 r P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + \dots$$

$$\phi_2 = \frac{e}{R} + C + \frac{B_0}{r} + \frac{B_1 P_1(\cos\theta)}{r^2} + \frac{B_2 P_2(\cos\theta)}{r^3} + \dots \quad (11)$$

We must find values of A_0, A_1, . . . B_0, . . . C so that the conditions (i)-(vi) are satisfied. Now (11) automatically satisfies (i), (ii) and (iv). It also satisfies (iii) if $C = 0$ and $B_0 = 0$. To deal with (v) and (vi) we have to use the fact that if $r < b$,

$$\frac{1}{R} = \frac{1}{b} + \frac{r}{b^2} P_1(\cos\theta) + \frac{r^2}{b^3} P_2(\cos\theta) + \dots \quad (12)$$

Conditions (v) and (vi) now give us two equations in θ, which must be satisfied for all values of θ. Since the P_n functions are linearly independent, we may equate coefficients of each P_n $(\cos\theta)$ on the two sides of each identity. Then (v) gives

$$A_0 = \frac{e}{b},\ A_n a^n = \frac{B_n}{a^{n+1}} + \frac{ea^n}{b^{n+1}},\ (n = 1, 2, \dots)$$

and (vi) gives

$$nKA_n a^{n-1} = -\frac{(n+1)B_n}{a^{n+2}} + \frac{nea^{n-1}}{b^{n+1}},\ (n = 1, 2, \dots).$$

In this way each coefficient A_n and B_n is obtained, and the potential is known.

If we want to find the force on the charge at B, we shall have to calculate the field there by differentiating ϕ_2. But the first term e/R in ϕ_2 must be omitted since this is simply the direct field of the charge and no charged particle can move itself by means of its own direct field. The effective field is therefore

$$-\frac{\partial}{\partial r}\left(\phi_2 - \frac{e}{R}\right)$$

measured at $\theta = 0$, $r = b$. Now at $\theta = 0$, each $P_n(\cos\theta) = 1$, so that the effective field is

$$\frac{2B_1}{b^3} + \frac{3B_2}{b^4} + \dots,$$

and the force on the charge at B is a force $\sum \frac{neB_{n-1}}{b^{n+1}}$ in the direction OB.

§ 79. A two-dimensional problem in magnetism

Problems in two dimensions are generally best done by the methods of our next chapter. But we can briefly illustrate how the methods of this chapter may also be applied, if we wish. Consider an infinitely long circular cylinder uniformly magnetised in a direction perpendicular to its axis (Fig. 52). Let us use cylindrical polar co-ordinates in which the origin is at O, the z axis is taken to be the axis of the cylinder and the $\theta = 0$ direction (or x axis) is parallel to the permanent magnetisation **M**. We have to solve the equations (see § 68):

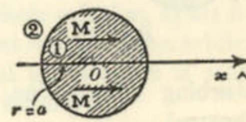


Fig. 52

(i) $\nabla^2\phi_1 = 0$, $\nabla^2\phi_2 = 0$,

(ii) $\phi_2 \to 0$ at infinity,

(iii) ϕ_1 is finite for $r \leqslant a$,

(iv) $\phi_1 = \phi_2$ on $r = a$,

(v) $-\frac{\partial\phi_1}{\partial r} + 4\pi M\cos\theta = -\frac{\partial\phi_2}{\partial r}$ on $r = a$. This is the condition that B_n is continuous at $r = a$.

Let us expand ϕ_1 and ϕ_2 in series form.

$$\phi_1 = \Sigma A_n r^n \cos n\theta,\ n = 1, 2, \dots$$

$$\phi_2 = \Sigma \frac{B_n}{r^n}\cos n\theta,\quad n = 1, 2, \dots \quad (13)$$

This automatically satisfies (i), (ii) and (iii). Further, (iv) and (v) are satisfied if and only if

$$A_1 = 2\pi M,\ B_1 = 2\pi Ma^2,\ A_n = B_n = 0 \text{ when } n \geqslant 2.$$

In this way we have obtained the complete solution.

§ 80. Another two-dimensional problem

A rather more difficult two-dimensional problem is shown in Fig. 53. An infinitely long line-charge of strength e (see § 12, question 15) at A is parallel to an infinitely long dielectric cylinder of radius a, and $OA = f$. We are to find the potential function at all points.

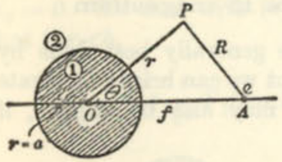


Fig. 53

The potential due to a line charge e at A, without any disturbing effects due to the dielectric, is (see previous reference)

$$\phi = -2e \log R + \text{constant},$$

where R is measured from the line charge A as in the figure. We have therefore to find two potential functions ϕ_1 and ϕ_2 valid in the regions 1 and 2, such that (see § 26):

(i) $\nabla^2\phi_1 = 0,\ \nabla^2\phi_2 = 0,$
(ii) ϕ_1 is finite in $r \leqslant a,$
(iii) $\phi_2 + 2e \log R$ is finite for $r \geqslant a,$
(iv) $\phi_1 = \phi_2$ on $r = a,$
(v) $K\dfrac{\partial \phi_1}{\partial r} = \dfrac{\partial \phi_2}{\partial r}$ on $r = a.$

We therefore put

$$\phi_1 = \sum_{n=0}^{\infty} A_n r^n \cos n\theta,$$

$$\phi_2 = -2e \log R + \sum_{n=0}^{\infty} \frac{B_n}{r^n} \cos n\theta \quad . \quad . \quad (14)$$

This automatically satisfies (i), (ii) and (iii). To satisfy (iv) and (v) we use the fact that if $r<f$,

$$\log R = \log f - \frac{r}{f}\cos\theta - \frac{r^2}{2f^2}\cos 2\theta - \ldots \quad . \quad (15)$$

This is most easily proved in complex co-ordinates by putting $z = x+iy$, expanding $\log(f-z)$ in powers of z/f, and then taking the real part. Thus condition (iv) gives, on comparing coefficients of $\cos n\theta$:

$$A_0 = -2e \log f + B_0,\ A_n a^n = \frac{2ea^n}{nf^n} + \frac{B_n}{a^n}.$$

In the same way condition (v) gives:

$$nKA_n a^{n-1} = \frac{2ea^{n-1}}{f^n} - \frac{nB_n}{a^{n+1}}. \quad (n = 1, 2, \ldots)$$

These two equations determine the whole solution, apart from an arbitrary constant. Such a constant represents an arbitrary zero of potential (in two-dimensional problems it is seldom possible to put the potential zero at infinity).

When solving such problems as these, in two or three dimensions, the student is strongly advised to follow the same procedure as in §§ 76-80, i.e. to begin by making a list of all the conditions that the potential must satisfy. It is impossible then to leave out any of the conditions by mistake, and the whole solution can be set out quite systematically.

§ 81. Problems with axial symmetry—a useful device

When a three-dimensional potential problem has axial symmetry, it is generally best to use spherical polar co-ordinates r, θ, ψ, and to take the axis of symmetry as the polar axis $\theta = 0$. In such a case the potential is independent of ψ, and we can write

$$\phi = \sum_n \left\{ A_n r^n + \frac{B_n}{r^{n+1}} \right\} P_n(\cos\theta), \quad . \quad . \quad (16)$$

where A_n and B_n are certain constants to be determined.

Along the axis of symmetry $\theta = 0$ so that $P_n(\cos\theta) = 1$, and $r = z$; thus ϕ reduces to

$$\phi_{axis} = \sum_n \left\{ A_n z^n + \frac{B_n}{z^{n+1}} \right\} \quad . \quad . \quad . \quad (17)$$

Now it may happen that we are able, by other means, to calculate the potential along the axis of symmetry. In that case, by comparing our result with (17) we determine at once the coefficients A_n and B_n. This means that the complete potential function (16) is now known, for points off the axis as well as for those on it.

The simplest illustration of this useful device is found in calculating the potential due to a circular wire carrying a current i. We have already determined in § 49 the potential at points along the axis of symmetry. Using the notation of Fig. 34 we have shown in (11) of § 49 that

$$\phi_{axis} = 2\pi i\{1 - z/\sqrt{(a^2+z^2)}\}.$$

If we write this in the form required by (17) we need two separate expansions:

$$z<a: \phi_{axis} = 2\pi i\left\{1 - \frac{z}{a} + \frac{z^3}{2a^3} - \ldots\right\}$$

$$z>a: \phi_{axis} = 2\pi i\left\{\frac{a^2}{2z^2} - \frac{3a^4}{8z^4} + \ldots\right\}.$$

It follows that the potential ϕ is given at all points by an expression of the form (16), where,

if $\quad r<a, B_n = 0, A_0 = 2\pi i, A_2 = A_4 = \ldots = 0,$

$$A_1 = \frac{2\pi i}{a},\ A_{2n+1} = 2\pi i(-1)^{n+1}\frac{1.3\ \ldots\ (2n-1)}{2.4\ \ldots\ 2n}\frac{1}{a^{2n+1}},$$

and if $r>a$, $A_n = 0$, $B_0 = 0$, $B_2 = B_4 = \ldots = 0$,

$$B_1 = 2\pi i\frac{a^2}{2},\ B_{2n+1} = 2\pi i(-1)^n\frac{1.3\ \ldots\ (2n+1)}{2.4\ \ldots\ (2n+2)}a^{2n+2}. \quad (18)$$

From this it is a straightforward matter to calculate the field at all points and so we complete the discussion which was commenced in § 49.

§ 82. Examples

1. Show that the least charge Q that can be given to a conducting sphere of radius a so that when the system is placed in a uniform field E_0 no part of the sphere is negatively charged is $Q = 3a^2E_0$.

2. A conducting cylinder of radius a and infinite length is placed with its axis perpendicular to a uniform field E_0. The cylinder is at zero potential. Calculate the potential at all points and deduce that the greatest surface density of induced charge is $E_0/2\pi$.

3. Show that (8) gives for the field inside a small circular cavity in a dielectric the result stated at the end of § 31.

4. A charge e is placed at a point P outside an uncharged conducting sphere of radius a. The distance of P from the centre O of the sphere is f. Show that the potential at any point whose distance from P is R may be put in the form

$$\phi = \frac{e}{R} - e\sum_{n=1}^{\infty}\frac{a^{2n+1}}{f^{n+1}r^{n+1}}P_n(\cos\theta).$$

Show that this may be transformed to

$$\phi = \frac{e}{R} - \frac{ea/f}{R_1} + \frac{ea/f}{r},$$

where R_1 is the distance from the inverse point of P with respect to the sphere. (Charges ea/f at O and $-ea/f$ at the inverse point are called the images of P in the sphere: see next chapter.)

5. Verify that the extra terms in (9) are not needed, i.e. $A_2 = A_3 = \ldots = B_2 = B_3 = \ldots = 0$.

6. A point charge e is placed a small distance c from the centre of a spherical cavity of radius a in an infinite dielectric. Show that the charge experiences a force approximately equal to $\dfrac{2(K-1)}{2K+1}\dfrac{e^2c}{a^3}$ away from O.

7. A spherical hole of radius a is cut out of an infinite block of uniform conductivity. At large distances the current flow is

uniform and in the z direction. Show that the potential function is $\phi = A\left(r + \frac{a^3}{2r^2}\right)\cos\theta$. Deduce that the lines of flow are $r^3 - a^3 = Cr\,\mathrm{cosec}^2\theta$.

8. Solve the potential equations for a uniformly magnetised sphere (cf. § 71).

9. A sphere of radius a and permeability μ is placed in a uniform magnetic field H_0. Prove that the field inside the sphere is uniform with value $3H_0/(\mu+2)$.

10. Verify that the solution in (10) for the potential due to a small magnet in a spherical cavity satisfies all the conditions (i)-(v) of § 77.

11. A magnetic particle of moment m is at the centre of a spherical hole of radius a cut in an infinite block of matter of permeability μ. A uniform field H_0 exists at infinity, and the magnetic particle points at an angle α with the direction of H_0. Show that the field inside the cavity consists of the field due to the magnet m together with a constant part whose direction makes an angle β with H_0, where

$$\cot\beta = \cot\alpha + \frac{3\mu a^3 H_0}{2(\mu-1)m\sin\alpha}.$$

12. A small magnet of moment m is at the centre of a spherical shell bounded by spheres of radii a and b and filled with material of permeability μ. Show that outside the shell the field is the same as that of a magnet of moment m', where

$$9\mu b^3 m = m'\{(\mu+2)(2\mu+1)b^3 - 2(\mu-1)^2a^3\}.$$

13. A spherical uncharged conductor of radius a is surrounded by a dielectric whose outer boundary is a concentric sphere of radius b. It is placed in a uniform field E_0. Show that the total positive and negative charges induced on the sphere are $\pm 9Ka^2b^3E_0/4\{(K+2)b^3+(2K-2)a^3\}$.

14. The point charge e at B in Fig. 51 is replaced by a dipole of moment m pointing away from O. Show that the potential outside the sphere may be obtained from the solution (11) in the form

$$\phi = \frac{\mathbf{m}.\mathbf{R}}{R^3} + \frac{C_1P_1(\cos\theta)}{r^2} + \frac{C_2P_2(\cos\theta)}{r^3} + \ldots,$$

where $$C_n = \frac{m}{e}\frac{\partial B_n}{\partial b}.$$

15. A charge e is placed at a point C distance c from O, lying between two conducting spheres of radii a and b, whose centres are at O. The two spheres are put to earth. Show that between the spheres the potential may be expressed in the form

$$\phi = \frac{e}{R} + \Sigma\{A_nr^n + B_nr^{-(n+1)}\}P_n(\cos\theta),$$

where R is the distance from C, r and θ are measured from O and the line OC respectively, and

$$A_n = -\frac{e}{c^{n+1}}\left\{\frac{c^{2n+1}-a^{2n+1}}{b^{2n+1}-a^{2n+1}}\right\},\ B_n = \frac{-ea^{2n+1}}{c^{n+1}}\left\{\frac{b^{2n+1}-c^{2n+1}}{b^{2n+1}-a^{2n+1}}\right\}.$$

16. A line source e is placed at a distance f from the axis of an infinite conducting cylinder of radius a, which is kept at zero potential. Show, by reasoning similar to that used in § 80, that the potential is

$$\phi = -2e\left\{\log R + \sum_{n=1}^{\infty}\frac{a^{2n}}{nf^nr^n}\cos n\theta + C\log r + D\right\},$$

where C and D are constants. Verify that this may be written

$$\phi = -2e\left\{\log R - \log R' + \log\frac{r}{f}\right\},$$

where R' is the distance from the inverse of the given line source in the cylinder.

17. A total charge Q is spread evenly over the surface of a circular disc of radius a. Show that at points on the axis of symmetry, a distance x from the disc, the potential is

$$\phi_{axis} = \frac{2Q}{a^2}\{\sqrt{(x^2+a^2)} - x\}.$$

Hence show that at any point for which $r>a$, the potential is

$$\phi = 2Q\sum_{n=0}^{\infty}(-1)^n\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n+2)}\frac{a^{2n}}{r^{2n+1}}P_{2n}(\cos\theta).$$

Find a corresponding expression valid for $r<a$.

18. A cylindrical bar magnet of radius a and length $2l$ is uniformly magnetised along its length with intensity M. Show

that at a point on the axis a distance x from the centre of the magnet

$$\phi_{axis} = 2\pi M\{\sqrt{[a^2+(x-l)^2]}-\sqrt{[a^2+(x+l)^2]}+2l\}.$$

By expanding in powers of x show that at large distances the potential takes the form

$$\phi = \frac{2\pi M a^2 l}{r^2} P_1(\cos\theta) + \text{terms of order } r^{-4}.$$

Deduce that the magnet behaves like a single particle of moment $2\pi M a^2 l$.

19. Two circular loops of radii a and b are perpendicular to the line joining their centres, which are a distance c apart ($c>a\geqslant b$). Unit current flows in the coil a whose centre is O. Use (18) to determine the radial component of the field measured from O, and by integrating over a suitable spherical surface show that the coefficient of mutual induction of the two loops (§ 53) is

$$M_{12} = 2\pi \sum_{n=0}^{\infty} \frac{2(n+1)B_{2n+1}}{R^{2n+1}} \int_{\mu_0}^{1} P_{2n+1}(\mu)\, d\mu,$$

where $\mu = \cos\theta$, $R^2 = b^2+c^2$, $\mu_0 = c/R$, and B_{2n+1} is the same as in (18) with $i = 1$. If R is large show that this gives $M_{12} = \dfrac{2\pi^2 a^2 b^2}{R^3} + O(R^{-5})$. (Cf. Chapter VI, question 15.)

CHAPTER X

SPECIAL METHODS

§ 83. Uniqueness

We saw in Chapter IX how it was possible to solve the potential equation for several special cases. But the whole of potential theory would be irrelevant and useless unless we could be sure that the function we obtained was unique. So we shall now prove that there is only one potential function which satisfies all the requisite conditions. The proof below applies to electrostatics, but the student will have no difficulty in adapting it to suit either current-flow or magnetostatic problems.

Imagine then that we have an electrostatic system consisting of one set of conductors, each carrying a given total charge: another set of conductors each kept at a given potential: and a given volume distribution of charge, in the presence of given dielectrics. This is the most general type of electrostatic problem. We shall show that there cannot be more than one potential function ϕ for this system.

For suppose that there are two potential functions ϕ_1 and ϕ_2, which satisfy all the required conditions. According to § 26 these are:

div (K grad ϕ_1) $= -4\pi\rho =$ div (K grad ϕ_2),
ϕ_1 and $\phi_2 \to 0$ at infinity,
ϕ_1 and ϕ_2 are finite except at point charges,
ϕ_1 and ϕ_2 are constant over the surface of each conductor, and have prescribed values on one set of conductors,

$\int K \frac{\partial \phi_1}{\partial n} dS = \int K \frac{\partial \phi_2}{\partial n} dS$, the integration being over each of the conductors whose total charge is given.

Put $\phi = \phi_1 - \phi_2$. Then in order to prove that ϕ_1 and ϕ_2 are identical, it will be sufficient to show that $\phi = 0$. Now from the separate equations for ϕ_1 and ϕ_2 it follows at once that

(i) $\operatorname{div}(K \operatorname{grad} \phi) = 0$,

(ii) $\phi \to 0$ at infinity,

(iii) ϕ is finite except at point charges,

(iv) ϕ = constant on the surface of all conductors, and in particular $\phi = 0$ on any conductor whose potential is prescribed,

(v) $\int K \frac{\partial \phi}{\partial n} dS = 0$ on any conductor whose total charge is given.

Now consider the integral $\int K \operatorname{grad} \phi . \operatorname{grad} \phi \, dv$, taken over all space outside the conductors, point charges counting as tiny spherical conductors. The integral transforms as follows:

$$\int K \operatorname{grad}\phi . \operatorname{grad}\phi \, dv = \int \{\operatorname{div}(\phi K \operatorname{grad}\phi) - \phi \operatorname{div}(K \operatorname{grad}\phi)\} dv$$

$$= \int \phi K \operatorname{grad}\phi . \mathbf{dS} - 0, \text{ from (i).}$$

This latter integral is taken over the sphere at infinity, where $\mathbf{dS}$ is determined by the outward normal, and over all the conducting surfaces where $\mathbf{dS}$ is determined by the normal directed into the surface. On each conductor whose potential is fixed, $\phi = 0$ from (iv), and the integral is zero. Over any of the other conductors whose charge is fixed, we can write

$$\int \phi K \operatorname{grad} \phi . \mathbf{dS} = -\phi \int K \frac{\partial \phi}{\partial n} dS = 0, \text{ from (v).}$$

Finally, at infinity ϕ vanishes at least to order $1/r$, so that $\operatorname{grad} \phi$ is at least of order $1/r^2$. Thus the integral over the sphere at infinity is also zero. Hence in all cases the integral vanishes and it follows that

$$\int K (\operatorname{grad} \phi)^2 \, dv = 0.$$

Since K is essentially positive, this means that $\operatorname{grad} \phi = 0$, so that $\phi_1 - \phi_2 \equiv \phi$ = constant. Over some of the surfaces, and at infinity, $\phi_1 = \phi_2$, and so the constant is zero. Hence ϕ_1 and ϕ_2 are identical, and the potential function is unique.

§ 84. An application of uniqueness

One application of this theorem may be given. Suppose that with a given system of charges we select two of the equipotential surfaces ϕ_1 and ϕ_2 completely surrounding all the charge. Let us make them into two conducting surfaces connected to the terminals of a battery. The potential distribution between these conductors is just the same as in the original field, provided that their potentials are kept at ϕ_1 and ϕ_2. The charges induced on the inner faces of the two conductors are $\pm \int \frac{K}{4\pi} \frac{\partial \phi}{\partial n} dS$. In particular, if we regard the two conductors as forming a condenser, all other charges being removed, its capacity is

$$\frac{1}{(\phi_1 - \phi_2)} \int \frac{K}{4\pi} \frac{\partial \phi}{\partial n} dS \quad . \quad . \quad . \quad (1)$$

§ 85. Images in a plane

A second very important application of the uniqueness theorem is found in the **method of images**. We can illustrate this by an example. Consider (Fig. 54) two charges $\pm e$ at A and B in free space. The lines of force go across from A to B as shown in the diagram, and the potential at any

point is $\phi = e/r_1 - e/r_2$, where r_1 and r_2 are the distances from A and B respectively. It is quite clear that along the mid-plane XY where $r_1 = r_2$, the potential is zero. In fact we may say that

$$\phi = e/r_1 - e/r_2 \quad . \quad . \quad . \quad . \quad (2)$$

is a potential function which gives potential zero all over the infinite plane XY, tends to infinity like e/r at A, and tends to zero at large distances. But these are precisely the conditions that must be satisfied by the potential when a point charge e is placed at A outside an infinite conducting

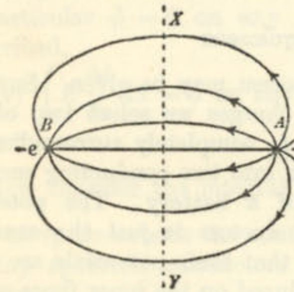


FIG. 54

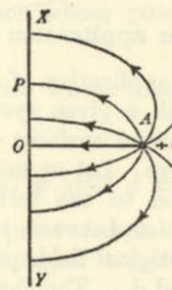


FIG. 55

plane XY at zero potential (Fig. 55). So by the principle of uniqueness we have now solved this latter problem. It follows, therefore, that on the right-hand side of XY the potential is given by (2).

We may describe the situation by saying that if a charge e is at A outside the infinite conducting plane XY which is kept at zero potential, the potential on the right of XY is just the same as if the plane were removed and replaced by a charge $-e$ at B. We refer to this as the **image charge** of A in the plane.

The potential (2) is, of course, only valid on the right of XY, and the lines of force which start from A end at XY, as shown in Fig. 55. Unless there are some other charges

present, the potential is zero everywhere to the left of XY. It is important to realise that the charge $-e$ at B has no more real existence than an optical image; all that we can say is that on the right of XY the system behaves *as if* we replaced the infinite plane by a charge $-e$ at B.

If we know the potential we can easily calculate the field and the density of charge at any point of XY. Thus at P in Fig. 55,

$$4\pi\sigma = -\frac{\partial\phi}{\partial n},$$

and on substituting the appropriate value of ϕ it is soon verified that

$$\sigma = \frac{-ae}{2\pi(AP)^3}, \quad . \quad . \quad . \quad . \quad (3)$$

where $a = AO$ is the shortest distance from A to the plane. The induced charge on XY is practically all concentrated near O and the density falls off as the inverse cube of AP.

Also, the force acting on the charge e at A is simply the force that would be exerted by the image charge $-e$ at B. This is an attraction towards the plane of magnitude $e^2/4a^2$.

The method of images is readily extended. Thus, let OX and OY (Fig. 56) be two semi-infinite perpendicular planes at zero potential, and let a charge $+e$ be placed at A. The student will easily verify that the image system consists of the original charge $+e$ at A, together with $-e$, $-e$ and $+e$ at B, C, D respectively. In the first quadrant XOY the potential at a point P is, again taking $K = 1$,

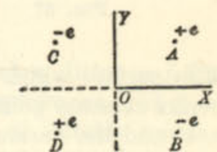


FIG. 56

$$\phi = e\left\{\frac{1}{AP} - \frac{1}{BP} - \frac{1}{CP} + \frac{1}{DP}\right\} \quad . \quad . \quad (4)$$

A similar method may be used if the angle XOY is of the

form π/n, where n is an integer. The case in which n is not an integer cannot be solved by the method of images (see § 90, question (25) for the corresponding problem in two dimensions), and we must have recourse to the technique of Chapter IX.

It is evident that an image system similar to that of Figs. 54-56 would exist if the point charge at A was replaced by a line charge e parallel to the planes. Thus, corresponding to the example of Fig. 55, we should have a line charge e parallel to an infinite conducting plane at zero potential. The potential on the right of XY would be the same as that due to two line charges $\pm e$ at A and B, so that

$$\phi = -2e \log r_1 + 2e \log r_2 \quad . \quad . \quad (5)$$

§ 86. Images with spheres and cylinders

Another important image system is found as follows. Consider (Fig. 57) two charges $+e$ and $-e'$ at A and B. The potential due to these by themselves is

$$\phi = e/r_1 - e'/r_2,$$

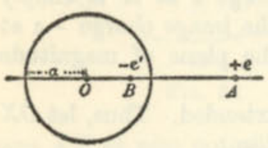


FIG. 57

where r_1 and r_2 denote distances from A and B respectively. So $\phi = 0$ on the surface

$$r_1/r_2 = e/e' \quad . \quad . \quad (6)$$

This surface which is spherical is shown in Fig. 57 with its centre at some point O on the line AB. The relation between e, e' and the various distances may be put in a more useful form by letting $OA = f$, and the radius of the sphere be a. We leave it as an exercise to the student to verify that (6) is equivalent to the following statement—a charge e at a point A distance f from the centre of a sphere of radius a, together with a charge $-ea/f$ $(= e')$ at a point B on the line OA such that $OB = a^2/f$, give zero potential on the sphere. A and B are obviously inverse points with respect to the sphere.

It follows, just as in § 85, that if a charge $+e$ is placed at A outside a conducting sphere connected to earth, then the potential outside the sphere is the same as if we replaced the sphere by a charge $-ea/f$ at the inverse point B. Thus, if r_1 and r_2 have the same meaning as in (6), the potential outside the sphere is

$$\phi = \frac{e}{r_1} - \frac{ea/f}{r_2} \quad . \quad . \quad . \quad (7)$$

Inside the sphere the potential is zero. The charge $-ea/f$ at B is evidently the image charge of e at A.

Once we know the potential all other quantities are soon found. In particular, since tubes of force can only begin and end on charges, Fig. 57 shows us that the net number of tubes that arrive on the sphere's surface is simply the same as the number that would end on the image charge at B in the absence of the sphere. Thus the sphere carries a net negative charge $-ea/f$.

If, instead of being at zero potential the sphere is given a total charge Q, the complete potential will be the superposition of (7) and the potential due to an isolated sphere carrying charge $Q + ea/f$. In that case, measuring r from the centre O:

$$\phi = \frac{e}{r_1} - \frac{ea/f}{r_2} + \frac{(Q + ea/f)}{r} \quad . \quad . \quad (8)$$

In particular, if a charge e is placed at a distance f from the centre of an uncharged conducting sphere, so that $Q = 0$ in the above equation; the potential at outside points is the same as that due to the original charge, together with a charge $-ea/f$ at the inverse point and a charge $+ea/f$ at the centre of the sphere. (Cf. Chapter IX, question 4.) The latter two charges are now the images of the first charge.

A similar system of images can be calculated for the two-dimensional case of a line charge parallel to a conducting cylinder. We leave this as an exercise for the student.

By combining two or more sets of images, we can similarly discuss more complicated problems. Thus, since a dipole may be regarded as two large almost coincident charges, its image in a plane is a similar dipole and in a sphere it is a dipole at the inverse point, whose magnitude and direction are soon determined, together with a charge at the inverse point (§ **90**, question 11). Other examples of images will be found in the questions at the end of the chapter. The student will recognise that the method of images is often a very quick and useful way of short-circuiting a more elaborate solution of the potential equation.

§ 87. Two-dimensional problems, conjugate functions *

It often happens that the potential problem which we have to solve involves only two variables x, y. We have already met this situation in the flow of current in a thin plate (§ **39**) and in the magnetic field round a long straight wire (§ **51**). We shall devote the rest of this chapter to a discussion of such cases, using the powerful method of complex variables.

Our fundamental problem is to seek solutions of Laplace's equation, and then to make them satisfy the relevant boundary conditions. In the two-dimensional case that we are considering, Laplace's equation is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad . \quad . \quad . \quad (9)$$

In order to solve this equation we put $z = x+iy$, where $i = \sqrt{-1}$. The student must not confuse this complex variable z with the third space co-ordinate, which will indeed be completely absent till Chapter XI.

* Before reading further the student should familiarise himself with the properties of complex variables and conformal representation, as e.g. Phillips, *Complex Variable*, Oliver and Boyd Ltd. Readers not familiar with these subjects should omit the rest of the chapter, which will not be needed in the remainder of this book.

Now consider any function of z, which we write

$$w = f(z) = f(x+iy) = \phi(x, y) + i\psi(x, y), \quad . \quad (10)$$

so that ϕ and ψ are the real and imaginary parts of $f(z)$. Clearly by simple partial differentiation of $w = f(x+iy)$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0.$$

Separating the real and imaginary parts

$$\nabla^2\phi = 0, \ \nabla^2\psi = 0 \quad . \quad . \quad . \quad (11)$$

Hence ϕ and ψ, as defined by (10), are each possible potential functions satisfying (9). Their close relationship in (10) suggests that we call them **conjugate functions**. By partial differentiation of (10) we have

$$\frac{\partial w}{\partial y} = if'(z) = i\frac{\partial w}{\partial x}.$$

Equating the real and imaginary parts, we obtain the **Cauchy relations**:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = \text{real part of } f'(z),$$

$$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} = \text{imaginary part of } f'(z) \quad . \quad . \quad (12)$$

These two statements may be summed up in the one statement that if s_1 and s_2 are perpendicular directions related in the anti-clockwise fashion of Fig. 58, then

$$\frac{\partial\phi}{\partial s_1} = \frac{\partial\psi}{\partial s_2} \quad . \quad . \quad (13)$$

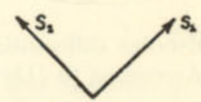


FIG. 58

In particular, if s_1 is taken along a contour $\phi = $ constant, $\partial\phi/\partial s_1 = 0$, so that $\partial\psi/\partial s_2 = 0$, showing that ψ is constant along the s_2 direction. Now for different values of the constant, $\phi(x, y) =$ constant represents a whole family

of curves. Thus $\phi(x, y) =$ constant, and $\psi(x, y) =$ constant, are two families of curves which cut orthogonally.

It is usual to regard ϕ as the potential function, and then ψ is called the **stream function**, though we could of course interchange ϕ and ψ.* The contours $\phi =$ constant are equipotentials, so that $\psi =$ constant must be the equation of the lines of force. In hydrodynamics $\psi =$ constant represents the lines of flow, and this accounts for ψ being called the stream function.

An example will illustrate what we have been saying. Let us take $f(z) = z^2$, so that (10) becomes

$$w = z^2 = x^2 - y^2 + 2ixy.$$

i.e. $$\phi(x, y) = x^2 - y^2, \ \psi(x, y) = 2xy.$$

We see that $\phi = 0$ on the two perpendicular lines $y = \pm x$; the other equipotentials are the rectangular hyperbolæ $x^2 - y^2 =$ constant, shown in Fig. 59. The lines of force are the orthogonal hyperbolæ $xy =$ constant, shown dotted in the diagram. This potential function would solve the problem of a condenser formed by two conductors, one of which is the planes $y = \pm x$, and the other is any member (e.g. *PQRS*) of the family of hyperbolæ $x^2 - y^2 =$ constant.

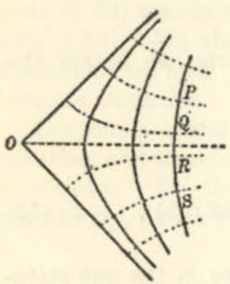


FIG. 59

It is not difficult to calculate the charge distribution in such a problem. For if σ is the density of charge at a point Q, $4\pi\sigma = -\partial\phi/\partial n$ where $\partial/\partial n$ denotes differentiation along the normal from right to left. According to (13) this is

$$4\pi\sigma = +\frac{\partial\psi}{\partial s}, \qquad (14)$$

where $\partial/\partial s$ denotes differentiation along the conductor from

* Some writers do interchange ϕ and ψ, and then they write $w = f(z) = \psi(x, y) + i\phi(x, y)$.

R to Q. Hence the total charge between Q and R per unit thickness perpendicular to the xy plane is q, where

$$q = \int \sigma\, ds = \frac{1}{4\pi}(\psi_Q - \psi_R) \qquad (15)$$

This gives us a ready means of calculating capacities.

We can also calculate the field **E**, since, from (12)

$$E_x = -\partial\phi/\partial x = -\text{ real part of } f'(z),$$
$$E_y = -\partial\phi/\partial y = \text{imaginary part of } f'(z) \qquad (16)$$

From this it follows that

$$E = |f'(z)| = \left|\frac{dw}{dz}\right|, \qquad (17)$$

and the direction that **E** makes with the x axis is

$$-\arg f'(z) = \pi - \arg\frac{dw}{dz}. \qquad (18)$$

These last two equations suggest that it is often easier to work directly in terms of w rather than in terms of ϕ and ψ. This is indeed the case, as can be seen from the following special cases.

(*a*) *Line charge e at the origin.*—We have already seen (Chapter II, question (15), and Chapter IX) that the potential ϕ for a line charge e at the origin is $\phi = -2e \log r$. This is obviously the real part of the function $-2e \log z$, so that

$$\phi = -2e \log r, \ \psi = -2e\,\theta, \ w = -2e \log z \qquad (19)$$

If the line charge is at the point z_0, then by a simple change of origin

$$w = -2e \log (z - z_0) \qquad (20)$$

(*b*) *Line electrode of strength I.*—For a line electrode in a medium of conductivity σ from which a current I* per unit

* In this chapter we must use I, rather than i, to denote the total current flowing along a wire or leaving an electrode, for though j is often used to denote $\sqrt{-1}$ in electrical current problems such as those of Chapter XII, the notation $z = x + iy$ is firmly established in mathematical tradition.

thickness is flowing, it follows from (20) and § 38 that

$$w = -\frac{I}{2\pi\sigma}\log(z-z_0) \quad . \quad . \quad . \quad (21)$$

(c) *Line doublet m at the origin.*—If the doublet is in the x direction, then by regarding it as the superposition of two large adjacent line charges, it can be seen that

$$\phi = \frac{2mx}{r^2},\ \psi = -\frac{2my}{r^2},\ w = \frac{2m}{z} \quad . \quad . \quad (22)$$

If the doublet is in the y direction

$$\phi = \frac{2my}{r^2},\ \psi = \frac{2mx}{r^2},\ w = \frac{2im}{z} \quad . \quad . \quad (23)$$

If the doublet makes an angle α with the x axis,

$$w = \frac{2me^{i\alpha}}{z}, \quad . \quad . \quad . \quad . \quad (24)$$

and if it is at the point $z = z_0$, then

$$w = \frac{2me^{i\alpha}}{z-z_0}. \quad . \quad . \quad . \quad . \quad (25)$$

(d) *Magnetic field of uniform current I in a straight wire.*—We have seen in § 51, equation (20) that if the current flows through the origin perpendicular to the xy plane, then the magnetostatic potential is $-2I\theta$. This is the real part of $2iI \log z$, so that

$$\phi = -2I\theta,\ \psi = 2I\log r,\ w = 2iI\log z \quad . \quad (26)$$

If the current flows through the point z_0,

$$w = 2iI\log(z-z_0) \quad . \quad . \quad . \quad (27)$$

(e) *Uniform field E_0.*—With a uniform field E_0 in the direction Ox

$$\phi = -E_0x,\ \psi = -E_0y,\ w = -E_0z \quad . \quad (28)$$

With a field E_0 in the direction Oy,

$$\phi = -E_0y,\ \psi = +E_0x,\ w = iE_0z \quad . \quad . \quad (29)$$

and if the field makes an angle α with the x axis,

$$w = -E_0e^{-i\alpha}z \quad . \quad . \quad . \quad . \quad (30)$$

Other cases will doubtless occur to the reader, but by suitable combinations of (19)-(30) it is possible to solve many problems of this nature. We give one example.

A line source e is at the point z_0 above the infinite plane $y = 0$, which is kept at zero potential. By the method of images (see (5)) we know that the potential is the same as that due to a line charge e at z_0 and a line charge $-e$ at the image point. This is the complex conjugate $\bar{z}_0$. Hence from (5) and (20),

$$\begin{aligned} w &= -2e\log(z-z_0)+2e\log(z-\bar{z}_0) \\ &= -2e\log\frac{z-z_0}{z-\bar{z}_0} \quad . \quad . \quad . \quad . \quad (31) \end{aligned}$$

From this it is easy to calculate the field and charge distribution.

§ 88. Conformal representation

The method of complex variable can sometimes be used very effectively in transforming one problem into another, which can then be solved. To do this we consider the transformation

$$\zeta = f(z), \quad . \quad . \quad . \quad (32)$$

which transforms a point $z = x+iy$ in the z-plane into a point $\zeta = \xi+i\eta$ in the ζ-plane.* We describe this as "mapping" the z-plane on to the ζ-plane. Since $d\zeta = f'(z)\,dz$, it follows that

$$|d\zeta| = |f'(z)|\,|dz|,\ \arg d\zeta = \arg f'(z)+\arg dz.$$

* See e.g. Phillips, *Complex Variable*, Chapter II.

Thus in the immediate neighbourhood of $z = z_0$, where $\zeta = \zeta_0 = f(z_0)$, all distances in the ζ-plane are $|f'(z_0)|$ times as large as the corresponding distances in the z-plane, and small arcs are turned through an angle arg $f'(z_0)$. Any small element of area in the z-plane becomes an element of area in the ζ-plane having the same shape as before but whose dimensions are each $|f'(z_0)|$ as great, and which is turned through an angle arg $f'(z_0)$. For this reason the process is described as **conformal representation.**

The importance of the transformation $\zeta = f(z)$ is that if we have any function $\phi(x, y)$ which in terms of ξ and η may be written $\Phi(\xi,\eta)$ then

$$\frac{\partial^2\Phi}{\partial\xi^2} + \frac{\partial^2\Phi}{\partial\eta^2} = M^2\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right\}, \quad . \quad . \quad (33)$$

where M, which is called the modulus of the transformation, is defined by

$$M = 1 \Bigg/ \left|\frac{d\zeta}{dz}\right| = \left|\frac{dz}{d\zeta}\right| \quad . \quad . \quad . \quad (34)$$

Thus if ϕ is a potential function in xy space, so that $\nabla^2\phi = 0$, then $\dfrac{\partial^2\Phi}{\partial\xi^2} + \dfrac{\partial^2\Phi}{\partial\eta^2} = 0$, so that Φ is a potential function in $\xi\eta$ space. In other words, potential functions transform into potential functions. Also, any curve or boundary along which ϕ is constant becomes a new curve or boundary along which Φ is constant.

This type of transformation is particularly useful because, if there is a line charge e at a point z_0, then in the transformed problem there is an equal line charge e at the related point ζ_0 where $\zeta_0 = f(z_0)$. For in the neighbourhood of z_0 and ζ_0,

$$\begin{aligned}\Phi(\xi,\eta) = \phi(x, y) &= -2e \log |z - z_0| + \text{terms finite at } z = z_0,\\ &= -2e \log \left|\frac{dz}{d\zeta}(\zeta - \zeta_0)\right| + \text{terms finite at } \zeta = \zeta_0,\\ &= -2e \log |\zeta - \zeta_0| + \text{terms finite at } \zeta = \zeta_0.\end{aligned}$$

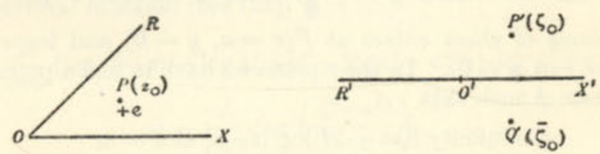
This result indicates that the potential corresponds to a line charge e in the ζ-plane at the transformed point ζ_0.

We can combine together the results of this paragraph by saying that the potential function for given boundaries and charges in the z-plane is precisely equivalent to the potential function for the transformed boundaries and charges in the ζ-plane. If we can solve this latter problem, then by transforming back to the z-plane we are able to solve the original problem. The only difficulties that may arise are connected with the zeros of M and $1/M$. If points z_0 occur in the region of interest for which $M = 0$ or $M =$ infinity, it will be necessary to investigate more closely the nature of the transformation near these points. It is often possible to avoid such singularities and we shall not discuss them further.

In more complicated problems it is sometimes an advantage to apply two or more conformal transformations successively. In this way we can simplify the problem stage by stage until it can finally be solved.

§ 89. Two worked examples

We conclude this chapter with two worked examples. Consider first (Fig. 60a) the problem of a line charge e at a point P where $z = z_0$, placed parallel to two infinite planes



(a) z-plane FIG. 60 (b) ζ-plane

OX, OR. The planes, which are kept at zero potential, make an angle π/n with each other. Our analysis applies equally well for all values of n, so that it completes the discussion of this problem that could be given by the method of images,

as in § 85, where it was necessary to suppose that n was an integer.

Take O as origin and apply the transformation $\zeta = z^n$, using dashes to denote the transformed points. This transformation maps the space between OX and OR into the whole space above the axis $R'O'X'$ in the ζ-plane (Fig. 60b). The line charge e at P transforms to an equal line charge at P' given by $\zeta_0 = z_0{}^n$. The new problem is therefore that of a line charge e placed above an infinite plane at zero potential. This has already been solved in (5) by the method of images, and

$$\Phi(\xi,\eta) = -2e \log |\zeta-\zeta_0| + 2e \log |\zeta-\bar{\zeta}_0|.$$

Putting this back in terms of the z-plane, we see that the potential required for our original problem is

$$\phi(x, y) = -2e \log |z^n - z_0{}^n| + 2e \log |z^n - \bar{z}_0{}^n|.$$

In terms of the w-function (§ 87),

$$w = -2e \log \frac{z^n - z_0{}^n}{z^n - \bar{z}_0{}^n} \quad . \quad . \quad . \quad (35)$$

Our second example is the current distribution in a thin sheet (see Fig. 61a) of infinite length bounded by the lines AB, CD for which $y = \pm\frac{\pi}{2}$, when a current I per unit thickness of sheet enters at $P(x = a,\ y = 0)$ and leaves at $Q(x = -a,\ y = 0)$. In the z-plane we have to find a potential function ϕ such that

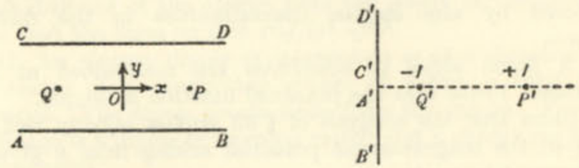
$$\phi \to \text{infinity like } -2I \log |z-a| \text{ at } z = a,$$
$$\phi \to \text{infinity like } -2I \log |z+a| \text{ at } z = -a,$$
$$\frac{\partial\phi}{\partial n} = 0, \text{ on } y = \pm\frac{\pi}{2} \text{ for all } x.$$

The transformation $\zeta = e^z$ is suited to this problem: for it gives

$$\xi = e^x \cos y, \quad \eta = e^x \sin y, \quad . \quad . \quad (36)$$

and the student will be able to verify that in passing from the z-plane to the ζ-plane (Fig. 61b)

(i) the line AB becomes $A'B'$, in which $\xi = 0$, $-\infty < \eta \leqslant 0$,
(ii) the line CD becomes $C'D'$, in which $\xi = 0$, $0 \leqslant \eta < +\infty$,
(iii) P and Q become P' and Q' where $\zeta_{P'} = e^a$, $\zeta_{Q'} = e^{-a}$,
(iv) the region between AB and CD becomes the half-plane $\xi > 0$.



(a) z-plane (b) ζ-plane
FIG. 61

Thus the corresponding problem that we must solve in the ζ-plane is that of a current I entering at P' and leaving at Q', with $\partial\Phi/\partial n = 0$ on the contour $B'D'$ where $\xi = 0$. We solve this by the method of images. The potential function in the relevant part of the ζ-plane is the same as that due to $+I$ at P', $-I$ at Q', together with image sources of current $+I$ at $\zeta = -e^a$, and $-I$ at $\zeta = -e^{-a}$; these last two points are the mirror images of P' and Q' in the line $\xi = 0$. So the potential function (see (21)) is

$$w = -\frac{I}{2\pi\sigma}\{\log(\zeta - e^a) + \log(\zeta + e^a) - \log(\zeta - e^{-a}) - \log(\zeta + e^{-a})\}$$
$$= -\frac{I}{2\pi\sigma}\log\left(\frac{\zeta^2 - e^{2a}}{\zeta^2 - e^{-2a}}\right).$$

In terms of the z-plane co-ordinates, the solution of our original problem is given by

$$w = -\frac{I}{2\pi\sigma}\log\left(\frac{e^{2z} - e^{2a}}{e^{2z} - e^{-2a}}\right). \quad . \quad . \quad (37)$$

In this way the flow of current is completely determined.

Since in the neighbourhood of the two points $z = \pm a$, the equipotentials are small circles, this potential function also solves the problem when the electrodes at P and Q are small circles instead of mathematical points.

§ 90. Examples

1. Show that the result of the uniqueness theorem in § 83 is unaffected by any sudden discontinuities in the dielectric constant.

2. A given series of electrodes are maintained at given potentials. Prove that the potential function is unique.

3. Show that the analysis of § 83 applies equally well with regard to the magnetostatic potential arising from a given set of permanent magnets.

4. S is a closed equipotential surface completely surrounding a system of charges whose algebraic sum is Q. The potential of S is V. Show that if the surface S is regarded as one plate of a condenser, of which infinity is the other, the capacity is Q/V. If S_1 and S_2 are two such surfaces at potentials V_1 and V_2, show that the capacity of the condenser formed by S_1 and S_2 is $Q/(V_1-V_2)$.

5. The space between two equipotential surfaces ϕ_1 and ϕ_2 which completely surround a system of electrostatic charges in vacuum is filled with a medium of constant dielectric constant K. Show that at all points inside the inner surface ϕ_1 the potential is increased by $\{(K-1)/K\}(\phi_1-\phi_2)$.

6. Show that if the angle XOY in Fig. 56 is made equal to π/n, where n is an integer, the number of charges in the image system of a charge at A inside XOY, is $2n$. Show also that unless n is an integer the method of images breaks down.

7. A line charge e is placed at a distance a from an infinite conducting plane at zero potential. Show that at a point P on the plane the density of induced charge is $-ea/\pi r^2$, where r is the shortest distance from P to the line charge.

8. Two equal charges e are placed a distance a apart, and each of them is $a/2$ from an infinite conducting plane at zero potential. Prove that the force between the charges is $3e^2/2a^2$. If the sign of one of the charges is reversed, why is the force not simply reversed also ?

9. The charge e_1 at A in Fig 57 is taken further away from the conducting sphere and simultaneously increased in size in such a way that as $f \to \infty$, $e_1/f^2 \to E_0$. Deduce that if an uncharged conducting sphere of radius a is placed in a uniform field E_0, the image of the field is a doublet at the centre of magnitude E_0a^3. This is the result of § 75.

10. A point charge e is placed inside a spherical cavity of radius a cut out of a conducting block of metal at zero potential. If the distance of the charge from the centre of the cavity is f, show that the force on it is $e^2af/(a^2-f^2)^2$.

11. An electric dipole of moment m is at a distance f from the centre of a conducting sphere of radius a kept at zero potential. The dipole points away from the sphere. Prove that the image is a dipole of moment ma^3/f^3 and a charge ma/f^2 at the inverse point.

12. An electric dipole of moment m is held at a distance a from an infinite conducting plane at zero potential. Show that the image is an equal dipole. If the original dipole makes an angle θ with the normal to the plane, show that the mutual potential energy is $-\dfrac{m^2}{8a^3}(1+\cos^2\theta)$. Deduce that if it is free to turn about its centre, it will take up a position perpendicular to the plane, and the period of small oscillations about this direction is $4\pi\sqrt{(2a^3A)}/m$, where A is the moment of inertia.

13. Combine the formulæ of §§ 85, 86 to solve the following problem. An infinite conducting plane with a hemispherical boss of radius a is kept at zero potential, and a charge e is placed on the axis of symmetry, a distance f from the plate. Show that the image consists of three charges, and that the original charge is attracted towards the plate with a force

$$\frac{4e^2f^3a^3}{(f^4-a^4)^2}+\frac{e^2}{4f^2}.$$

Show further, by the methods of § 11 that the lines of force which reach the plate at its junction with the hemisphere leave the charge e at an angle $\cos^{-1}\left\{1-\dfrac{2(f^2-a^2)}{f\sqrt{(f^2+a^2)}}\right\}$ with the axis of symmetry.

14. The whole of space for which $x>0$ is filled with a dielectric of uniform dielectric constant K, and the rest is vacuum. P

is the point $(-a, 0, 0)$ and P' is the point $(a, 0, 0)$. A charge e is placed at P. Show that in the vacuum the potential is the same as that due to a charge e at P and e' at P', and in the dielectric it is the same as that due to a charge e'' at P, where

$$e' = -\frac{K-1}{K+1}e, \quad e'' = \frac{2}{K+1}e.$$

Use the fact that the pull on the dielectric is equal and opposite to the pull on the charge e to show that the dielectric is attracted towards the charge with a force $\dfrac{K-1}{K+1}\dfrac{e^2}{4a^2}$. Explain this in terms of tensions in the tubes of force.

15. Show that the result in question (14) may be used to determine the image system when a small permanent magnet is held some distance away from an infinite block of material of permeability μ.

16. Show how to find the image system for a line charge parallel (i) to an infinite dielectric medium, and (ii) to an uncharged conducting cylinder at zero potential. By removing the line charge to infinity in (ii) show how to find the potential for a conducting cylinder in a uniform field perpendicular to its axis.

17. A current i flows in a straight wire parallel to a circular cylinder of permeability μ. Show that inside the cylinder the magnetic induction is $2\mu/(\mu+1)$ times as big as it would be if the cylinder were removed, and is everywhere in the same direction. Deduce that if the cylinder is placed in any given two-dimensional field, the induction inside it is unaffected in direction, but is multiplied by a factor $2\mu/(\mu+1)$ in magnitude.

18. A uniform current i flows in a straight wire parallel to a semi-infinite block of material of permeability μ. Show that the potential is given by an image system similar to that in question (14).

19. A telegraph wire is of radius a and its height above the earth is h. Prove by means of (5) that the capacity per unit length is $\{2 \log 2h/a\}^{-1}$.

20. A charge e is placed a distance f from the centre of an insulated conducting sphere of radius a. Show that the least positive charge which must be given to the sphere so that the surface density is everywhere positive, is $ea\{(f+a)/(f-a)^2-1/f\}$.

21. A and B are the points $z = \pm a$. Show that the function $w = \log \dfrac{z-a}{z+a}$ gives equipotential surfaces $r_1/r_2 =$ constant, where r_1 and r_2 are distances from A and B. Deduce that the capacity C per unit length between two parallel cylinders of radius R a distance $2D$ apart, is given by

$$1/C = 4 \log (\{D+\sqrt{(D^2-R^2)}\}/R) = 4 \cosh^{-1}(D/R).$$

22. A line charge e is at a distance h from an infinite conducting plane at zero potential. Show that one half of the charge induced on the plane lies within $\sqrt{2}h$ from the line charge.

23. Show that the capacity per unit thickness of a two-dimensional condenser in which the plates are at potentials V_1 and V_2 is $[\psi]/4\pi(V_1-V_2)$ where $[\psi]$ denotes the increment in ψ in going once completely round one of the plates.

24. Show that $w = -E_0\left(z - \dfrac{a^2}{z}\right)$ gives the potential distribution for a conducting cylinder of radius a at zero potential in a uniform field E_0. (Cf. Chapter IX, question 2.) Interpret the expression for w in terms of images.

25. Show that $w = iAz^n$ (n not necessarily integral) gives the potential distribution for two intersecting planes OX, OR making an angle π/n, if both are at zero potential. Prove that with this potential function, $\phi = -Ar^n \sin^n n\theta$. Find the equations of the lines of force, and deduce from (14) that the charge density at any point on the planes is $nAr^{n-1}/4\pi$. Draw diagrams for $n = 2$ and $n = \frac{2}{3}$, and notice the bunching of tubes of force near the tip in the one case, and the absence in the other. This bunching explains why a lightning conductor is given a pointed tip.

26. $ABCD$ is the region bounded on two sides AB, CD by the parabolas $y^2 = 4b(b-x)$, $y^2 = 4c(c-x)$, and on the other two sides AD, BC by the x axis and the parabola $y^2 = 4a(x+a)$. Show that the transformation $\zeta = z^{\frac{1}{2}}$ maps $ABCD$ on to a rectangle in the ζ-plane. Hence prove that if AB and CD are taken as electrodes, and if the strip is of uniform thickness t, its resistance is $(b^{\frac{1}{2}}-c^{\frac{1}{2}})/\sigma t a^{\frac{1}{2}}$.

27. Show that the total current leaving an electrode per unit thickness of material is $\sigma[\psi]$, where $[\psi]$ denotes the change in ψ on going round the boundary of the electrode in a clockwise direction.

28. A thin strip of metal of thickness t lies in the xy plane, and is bounded by the lines $y = \pm\frac{\pi}{2}$. Equal currents I enter and leave by small circular electrodes of radius δ at the points $(\pm a, 0)$. Show that the potential function w is

$$w = -\frac{I}{2\pi\sigma t}\log\left(\frac{e^{2z}-e^{2a}}{e^{2z}-e^{-2a}}\right) + \text{constant.}$$

Deduce that the resistance is $-\frac{1}{\pi\sigma t}\log(\delta \operatorname{cosech} 2a)$.

29. Show that if n is an integer, (35) may be interpreted as showing that if two planes at zero potential intersect at an angle π/n, the image system of a line charge e between them consists of $n-1$ line charges $+e$, and n line charges $-e$.

30. If the strip in Fig. 61a is of width $2b$, show that the transformation (36) needs to be replaced by $\zeta = e^{\pi z/2b}$, and that

$$w = -\frac{I}{2\pi\sigma}\log\left(\frac{e^{\pi z/b}-e^{\pi a/b}}{e^{\pi z/b}-e^{-\pi a/b}}\right).$$

31. A condenser is formed by taking OX, OR (Fig. 60a) as one plate, and a small circular cylinder with centre P and radius δ as the other. Show that its capacity per unit length perpendicular to the xy plane is $-1\Big/2\log\left\{\frac{n\delta}{2r}\operatorname{cosec} n\alpha\right\}$, where $z_0 = re^{i\alpha}$.

32. What happens if you use the transformation $\zeta = z^{2n}$ in § 89, Fig. 60a, instead of $\zeta = z^n$?

33. Show that in the transformation $\zeta = f(z)$ a line doublet of moment m at z_0, making an angle α with the x axis, transforms into a line doublet of moment $m|f'(z_0)|$ at $\zeta_0 = f(z_0)$, making an angle $\alpha + \arg f'(z_0)$ with the ξ axis. Deduce that if the original doublet lies between the two planes of Fig. 60a, which are kept at zero potential,

$$w = 2mn\left|z_0^{\,n-1}\right|\left\{\frac{e^{i\theta}}{z^n - z_0^{\,n}} - \frac{e^{-i\theta}}{z^n - \bar{z}_0^{\,n}}\right\},$$

where $\theta = \alpha + (n-1)\arg z_0$.

CHAPTER XI

INDUCTION

§ 91. Electromagnetic induction

In 1831 Michael Faraday, and independently at about the same time, Henry, discovered that if a closed circuit moved across a magnetic field, a current flowed even though there were no batteries present. The same effect resulted from varying the magnetic field and keeping the loop of wire still. In either case the current lasted only so long as the circuit was moving, or the field changing. It made no difference whether the magnetic field was caused by a permanent magnet or an electrical circuit. To this phenomenon Faraday gave the name **electromagnetic induction.** In a series of brilliant experiments he showed that the induced current flowed whenever the number of tubes of induction N through the circuit was altered, and that the magnitude of the effect depended only on the rate of change of N. It was natural that Faraday, having abandoned the idea of action-at-a-distance, should regard this effect as transmitted by means of the tubes of magnetic induction **B**.

We may summarise the experiments by two laws :—

(a) if the number N of tubes of induction threading a circuit is changed in any way, there is an electromotive force (e.m.f.) of magnitude dN/dt created in the circuit;

(b) this e.m.f. acts in such a way as to oppose the change in N. That is, it acts to cause currents in the coil whose magnetic effect would counteract the external change.

The first of these is known as **Neumann's law**; the second as **Lenz's law.** Provided that the e.m.f. and N

are both in e.m.u. or both in e.s.u., we can combine them in the one law

$$\text{e.m.f.} = -\frac{dN}{dt} \quad . \quad . \quad . \quad . \quad (1)$$

Lenz's law is merely a particular case of a very general physical principle, known as **Le Chatelier's Principle,** which states that a physical system always reacts to oppose any change that is imposed from outside. In this case the change is an alteration in N; the reaction is an induced e.m.f., or back e.m.f., which should oppose the change of N.

§ 92. Proof of induction law for a stationary circuit

Faraday obtained his results by experiment. But we can prove (1) by using laws with which we are already familiar. Thus let us first consider a single stationary isolated circuit with a battery of e.m.f. $\mathcal{E}$ carrying current i, with a resistance R and self-induction L. Then according to (38) of § 64 there is an amount of magnetic energy $\frac{1}{2}Li^2$ stored in the medium. This means that if we increase the flux through the circuit by changing i we have also to increase the magnetic energy, and hence that more work has to be done by the batteries. In more precise language the battery provides energy at a rate $\mathcal{E}i$; Joule heat loss wastes energy at a rate Ri^2; and we increase the energy in the medium at a rate $\frac{d}{dt}(\frac{1}{2}Li^2)$. Thus by the conservation of energy.

$$\mathcal{E}i = Ri^2 + \frac{d}{dt}(\tfrac{1}{2}Li^2)$$

So

$$\mathcal{E} = Ri + L\frac{di}{dt}. \quad . \quad . \quad . \quad (2)$$

During any period when the current is not changing

$$\mathcal{E} = Ri.$$

If we write (2) in the form

$$\mathcal{E} - L\frac{di}{dt} = Ri, . \quad . \quad . \quad . \quad (3)$$

we can interpret it by saying that the change in current gives an effective back e.m.f. of magnitude $L\frac{di}{dt}$, and Ohm's law is still obeyed provided that we regard the effective e.m.f. in the circuit as the sum of the battery e.m.f. $\mathcal{E}$ and the induced back e.m.f. $-L\frac{di}{dt}$. Since the flux of **B** across the circuit is $N = Li$, the back e.m.f. may be written $-\frac{dN}{dt}$, as in (1).

If this back e.m.f. is solely due to the change in N, it cannot make any difference how this change is brought about. Thus, if we move a permanent magnet in the neighbourhood of the coil there will be an additional change in N, and this will make its appropriate contribution to the induced e.m.f. This explains why Faraday was able to obtain a flow of current in the original coil when a permanent magnet was dropped through it. But we could also induce an e.m.f. in the original coil by varying the current in some nearby coil, making use of the fact (§ 60) that when current i_2 flows in the second coil a flux $N = M_{12}i_2$ crosses the first coil; in fact, an analysis exactly similar to that leading to (3) shows that if i_2 varies, the back e.m.f. in the first coil is $-\frac{dN}{dt} = -M_{12}\frac{di_2}{dt}$. We shall return to this again in § 96.

§ 93. Proof of induction law for a moving circuit

A simple consideration of the conservation of energy shows that since magnetic effects are transmitted by the tubes of force, it cannot make any difference whether we move the lines of **B** across the circuit, or move the circuit across the lines of **B**. The theorem must therefore hold for a moving

circuit as well as a stationary one. However, it is instructive to give an independent proof for the case of a moving circuit, when the field in which it moves does not vary with the time. Consider, therefore, an element PQ (Fig. 62) represented by $\mathbf{ds}$, which has velocity $\mathbf{v}_1$. In time dt it will move a distance $\mathbf{v}_1 dt$ to $P'Q'$. Suppose also that the velocity of the electrons relative to the wire is $\mathbf{v}_2$. If no current is flowing $\mathbf{v}_2 = 0$, but when current does flow the actual velocity of the electrons relative to the field is $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. Thus, from (30) in § 62, the magnetic field exerts on each electron a force

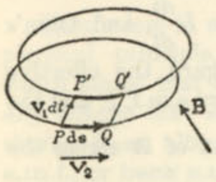


Fig. 62

$$e(\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{B}.$$

The component of this along PQ is $e\{(\mathbf{v}_1+\mathbf{v}_2)\times\mathbf{B}\}.\mathbf{t}$, where $\mathbf{t}$ is a unit vector in the direction PQ. Now $\mathbf{t}$ and $\mathbf{v}_2$ are parallel vectors, so that $(\mathbf{v}_2 \times \mathbf{B}).\mathbf{t} = 0$, and the component force along PQ is the triple product $e[\mathbf{v}_1, \mathbf{B}, \mathbf{t}]$.* This means that in addition to any forces due to possible batteries, there is an extra electric field induced in the wire, whose component along the wire is $[\mathbf{v}_1, \mathbf{B}, \mathbf{t}]$. The corresponding difference of potential between P and Q is

$$V_P - V_Q = [\mathbf{v}_1, \mathbf{B}, \mathbf{t}]\, ds.$$

Now $\mathbf{t}\, ds = \mathbf{ds}$, so that

$$V_P - V_Q = [\mathbf{v}_1, \mathbf{B}, \mathbf{ds}].$$

We have therefore shown that if a small piece of wire is moved in a magnetic field, there is a difference of potential between its ends. We can reduce this to the fundamental law of induction by putting

$$[\mathbf{v}_1, \mathbf{B}, \mathbf{ds}] = -[\mathbf{v}_1, \mathbf{ds}, \mathbf{B}] = -\frac{1}{dt}(\mathbf{v}_1 dt \times \mathbf{ds}).\mathbf{B}$$

* Rutherford, *Vector Methods*, 1946, p. 8.

Now $\mathbf{v}_1 dt \times \mathbf{ds}$ is the vector area $PP'Q'Q$, so that $(\mathbf{v}_1 dt \times \mathbf{ds}).\mathbf{B}$ represents the flux of $\mathbf{B}$ across this area. Thus

$$V_P - V_Q = -\frac{dN}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where dN is the number of tubes of $\mathbf{B}$ crossing into $PP'Q'Q$. Thus the induced e.m.f. in a small element PQ is equal to minus the rate at which PQ cuts across the tubes of $\mathbf{B}$. By adding together the induced e.m.f. for each element of the circuit, it follows that the total induced e.m.f. in a closed circuit is $-\frac{dN}{dt}$, where N now stands for the number of tubes of induction threading the circuit.

The above proof is interesting since it shows just how the induced e.m.f. is distributed over the circuit. It is not concentrated like a battery, but each moving element of the circuit contributes according to (4).

The nature of the proof also shows that it is quite irrelevant by what means N is changed. This may be done by moving a permanent magnet, or changing the current in some other nearby coil, or moving the original circuit, or by combining all three. In each case if $\mathbf{E}$ is the field at any point of the wire, and $\mathcal{E}_{\text{battery}}$ is the e.m.f. of any batteries that may be present, then the effective e.m.f. in the circuit is

$$\int \mathbf{E}.\mathbf{ds} = \mathcal{E}_{\text{battery}} - \frac{dN}{dt} \quad . \quad . \quad . \quad (5)$$

§ 94. A simple dynamo

This simple law of induction is the basis on which all dynamos, or machines for creating an e.m.f., are based. Consider, for example, an almost closed circular loop of wire PQ of area A (Fig. 63) which can rotate in air about an axis XY: and let us suppose that there is a constant magnetic field H perpendicular to the plane of the paper. Then when the normal to the loop makes an angle θ with the field, the flux $N = AH \cos\theta$, so that the e.m.f. induced between

P and Q when the coil is rotating with angular velocity p, is

$$\mathcal{E} = -\frac{dN}{dt} = -\frac{d}{dt}(AH\cos\theta) = AH\,p\sin\theta \quad . \quad (6)$$

If P and Q were joined together, completing the circuit, this e.m.f. would cause a current to flow round the wire. But if by some device such as brushes, or sliding contacts on the axle XY, we connect PQ to the terminals of an outside circuit, this e.m.f. may be used to create a current in that circuit. On account of the $\sin\theta$ term in (6) the e.m.f. is **alternating**, with frequency $p/2\pi$. A **direct** e.m.f. may be obtained by a **commutator** which interchanges the roles of P and Q in the outside circuit at every half-revolution. The e.m.f. thus obtained still fluctuates between 0 and AHp, and commercial dynamos therefore have a large number of similar coils, each slightly displaced relative to its neighbours, so that their **phases** are staggered. In this way the fluctuations are smoothed out, and the resulting combination may be made almost steady.

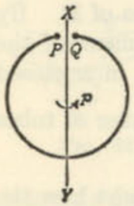


FIG. 63

If the air within the coil is replaced by a medium of permeability μ, the flux of **B** is increased by a factor μ so that the e.m.f. is

$$\mathcal{E} = \mu AHp\sin\theta \quad . \quad . \quad . \quad (7)$$

To obtain this effect the coils PQ are wound on an **armature** or **core** of soft iron. The field **H** may be obtained from a permanent magnet, but more frequently it is caused by an electromagnet, the coils of which are known as **field coils**, since they provide the field in which the armature rotates.

§ 95. Induction in a single circuit

Let us apply the law of induction (1) to a single circuit. Consider, for example, the circuit shown in Fig. 64, in which the resistance is R, the self-induction is L and the e.m.f.

of the battery is $\mathcal{E}$. According to (5), when the current is i, the effective forward e.m.f. is $\mathcal{E}-L\frac{di}{dt}$, so that Ohm's law gives the differential equation for i :—

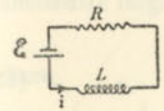


FIG. 64

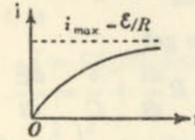


FIG. 65

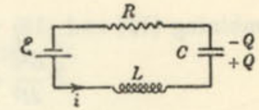


FIG. 66

$$\mathcal{E}-L\frac{di}{dt} = Ri \quad . \quad . \quad . \quad (8)$$

If we suppose that at $t = 0$ the circuit is closed by a switch so that at this instant $i = 0$ also, the solution of (8) is

$$i = \frac{\mathcal{E}}{R}(1-e^{-Rt/L}) \quad . \quad . \quad . \quad . \quad (9)$$

Thus the current builds up to its maximum value $\mathcal{E}/R$ according to the graph shown in Fig. 65. If we write $\tau = L/R$, $i_{\max} = \mathcal{E}/R$, then

$$i = i_{\max}(1-e^{-t/\tau}) \quad . \quad . \quad . \quad . \quad (10)$$

τ is called the **time constant** of the circuit.

Another important circuit is shown in Fig. 66. This is similar to that of Fig. 64 except for the addition of a condenser of capacity C. At time t, let i be the current and let $\pm Q$ be the charges on the plates of the condenser. The drop in potential across the condenser is Q/C. Hence the net forward e.m.f. is now

$$\mathcal{E}-L\frac{di}{dt}-\frac{Q}{C}.$$

So the differential equation for the current is

$$\mathcal{E}-L\frac{di}{dt}-\frac{Q}{C} = Ri \quad . \quad . \quad . \quad (11)$$

L, R and C are constants; but Q changes on account of the current. In fact, since the current is measured by the rate of flow of charge along the wire,

$$\frac{dQ}{dt} = i \quad . \quad . \quad . \quad . \quad (12)$$

Combining (11) and (12) we get

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{d\mathcal{E}}{dt}. \quad . \quad . \quad (13)$$

Two special cases of (13) are worth discussing in more detail :—

(*a*) *Free oscillations.*—In the first we put $\mathcal{E} = 0$, so that we are concerned only with the free oscillations of charge within the circuit. Equation (13) becomes

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0, \quad . \quad . \quad . \quad (14)$$

and provided that $R^2 < 4L/C$ the solution is

$$i = Ae^{-Rt/2L}\cos(nt+\epsilon) \quad . \quad . \quad . \quad (15)$$

A and ϵ are arbitrary constants depending only on the initial conditions, and

$$n^2 = \frac{1}{LC} - \frac{R^2}{4L^2}. \quad . \quad . \quad . \quad (16)$$

Thus there are oscillations of period $2\pi/n$, but the amplitude decays exponentially with the time according to the law $Ae^{-Rt/2L}$. If we can neglect the last term in (16), i.e. if the resistance is small, we obtain Thomson's formula for the period T of free oscillations,

$$T = 2\pi\sqrt{(LC)} \quad . \quad . \quad . \quad (17)$$

(*b*) *Forced oscillations.*—In our second case we suppose that $\mathcal{E} = \mathcal{E}_0 \cos pt$. Thus the applied e.m.f. is an alternating one of frequency $p/2\pi$. (13) becomes

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = -p\mathcal{E}_0 \sin pt \quad . \quad . \quad (18)$$

The solution of (18) involves the sum of two parts—the Complementary Function and Particular Integral.* The Complementary Function is simply (15) and represents free oscillations, or **transients**, that die away with the time. But the Particular Integral, which alone is important for sufficiently large t, is

$$i = (\mathcal{E}_0/Z)\cos(pt-\delta),$$

where

$$Z^2 = R^2 + \left(pL - \frac{1}{pC}\right)^2, \ \tan\delta = \left(pL - \frac{1}{pC}\right)\Big/ R. \quad . \quad (19)$$

This alternating current has the same frequency as the applied e.m.f. $\mathcal{E}_0 \sin pt$, and its amplitude $\mathcal{E}_0/Z$ is independent of the time. It is, however, out of phase with the e.m.f. We call this a forced oscillation. If all the other quantities remain constant, the amplitude of the forced oscillations has its greatest value when

$$p^2 = 1/LC \quad . \quad . \quad . \quad . \quad (20)$$

This is known as the **resonance frequency** of the circuit. We make use of this frequency when tuning the aerial circuit of a wireless receiver.

§ 96. Two circuits—the transformer

Suppose that we have two coils with self-inductances L_1, L_2 and mutual inductance M, which form parts of two circuits as in Fig. 67. On account of the mutual induction we describe these circuits as **inductively coupled**.

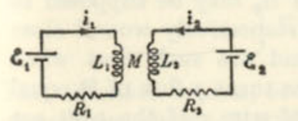


FIG. 67

Let i_1, i_2 be the currents in the two circuits. Then the flux of **B** through the first coil is $L_1i_1 + Mi_2$. When i_1 and i_2 change

* See e.g. Ince, *Differential Equations*, Chapter V.

this gives rise to a back e.m.f. $L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$. Consequently the net forward e.m.f. in this circuit is

$$\mathcal{E}_1 - L_1 \frac{di_1}{dt} - M \frac{di_2}{dt},$$

and Ohm's law now gives the differential equation

$$\mathcal{E}_1 - L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} = R_1 i_1 \quad . \quad . \quad (21)$$

Similarly from the other circuit

$$\mathcal{E}_2 - L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} = R_2 i_2 \quad . \quad . \quad (22)$$

Corresponding equations are soon written down in a similar way for any number of linked circuits. They are easily solved if $\mathcal{E}_1$, $\mathcal{E}_2$ . . . are known and the initial conditions are stated.

There is one particular case of (21) and (22) which is of especial interest. Let us suppose that $\mathcal{E}_1$ is an alternating e.m.f., and both $\mathcal{E}_2$ and R_1 are zero. This means that the current in the second circuit is solely due to the e.m.f. in the first. We shall see that so far as the second circuit is concerned, the effect of the mutual inductance is to provide an e.m.f. of a different numerical value from the original e.m.f. in the first circuit.

In practice the two coils L_1 and L_2 may be supposed to consist of n_1 and n_2 turns of wire respectively wound close together on the same iron core, and are such that when unit current flows in any one of these turns a flux of **B** equal to B_0 crosses this particular turn of wire. If the coils are wound closely, this flux will cross all the other turns in both coils. Thus unit current in the coil L_1 produces a total flux n_1B_0, and this flux threads each of the n_1 turns of the first coil, and each of the n_2 turns of the second. Hence,

from the definitions of L and M (see § 55, question (14), for further discussion of this result)

$$L_1 = n_1^2B_0,\ M = n_1n_2B_0,\ L_2 = n_2^2B_0 \quad . \quad (23)$$

It follows that

$$L_1L_2 = M^2. \quad . \quad . \quad . \quad (24)$$

This relation, sometimes called the **transformer condition,** will not be quite completely satisfied in practice on account of leakage of lines of **B**; but if we assume that it is satisfied and put $\mathcal{E}_2 = R_1 = 0$, we can eliminate i_1 from (21) and (22) and obtain

$$\mathcal{E}_1L_2 = -MR_2i_2 \quad . \quad . \quad . \quad (25)$$

Using (23) this becomes

$$R_2i_2 = -\frac{n_2}{n_1}\mathcal{E}_1 \quad . \quad . \quad . \quad (26)$$

This shows that the potential R_2i_2 across the ends of the second coil is in a constant ratio n_2/n_1 to the e.m.f. in the first coil. In this way, by varying n_1/n_2, we are able to step up or down a given alternating e.m.f. For this reason an apparatus of this kind is called a transformer.

§ 97. Generalised law of induction

We conclude this chapter with a remarkable generalisation of the law of induction, due to Maxwell. If we consider a stationary circuit in which there are no batteries, (5) becomes

$$\int \mathbf{E}.\mathbf{ds} = -\frac{dN}{dt}. \quad . \quad . \quad . \quad (27)$$

This equation applies, so far, only for integration round the closed wire circuit, **E** and **ds** both being directed along the wire. But the tangential component of **E** is continuous at the surface of the wire where the medium changes, and so (27) must also hold for a circuit lying just near to the wire. Indeed, we may expect it to hold along any closed curve, since the

fact that it holds for at least one curve not coincident with the wire shows that the electrical forces (represented by **E**) which are brought into existence by a change of magnetic flux cannot depend on whether there is an actual flow of current. The flow of current is here an "effect" and not a "cause"; and the induced or back e.m.f. is independent of whether a current does, or even can, flow. So let us assume the truth of (27) for any closed curve whatever. To some extent this is a hypothesis, but we have shown how reasonable it is, and it has been abundantly justified by the results which it predicts.

We may write $N = \int \mathbf{B}.d\mathbf{S}$, where the integration is over any surface that spans the curve along which **E** is integrated. Thus

$$\int \mathbf{E}.d\mathbf{s} = -\frac{\partial}{\partial t}\int \mathbf{B}.d\mathbf{S} = -\int\left(\frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}\right).$$

But by Stokes' theorem

$$\int \mathbf{E}.d\mathbf{s} = \int \text{curl}\,\mathbf{E}.d\mathbf{S},$$

and so for any surface S spanning an arbitrary contour:

$$\int \text{curl}\,\mathbf{E}.d\mathbf{S} = -\int\left(\frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}\right).$$

This is only possible if

$$\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad . \quad . \quad . \quad (28)$$

This is an extremely important formula, and may be called the generalised or differential form of the law of induction. Our former expression (5) is merely a particular case of (28) when the current distribution is along a thin wire and we choose this wire as the boundary curve for integrating (28).

All quantities above are supposed measured in e.m.u. But if, as is usual, we measure **E** in e.s.u. and **B** in e.m.u., it becomes

$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \quad . \quad . \quad . \quad (29)$$

We can go one stage further by introducing the magnetic vector potential **A**. Let us, as in § 58 (v), put $\mathbf{B} = \text{curl}\,\mathbf{A}$. Then

$$\text{curl}\left\{\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right\} = 0.$$

The expression in brackets must be the gradient of some scalar quantity. Let us call this $-\text{grad}\,\phi$. Then

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \text{grad}\,\phi \quad . \quad . \quad . \quad (30)$$

If there is no change of magnetic field $\frac{\partial \mathbf{A}}{\partial t} = 0$, so that $\mathbf{E} = -\text{grad}\,\phi$. This identifies ϕ in (30) with the usual electrostatic potential V which we have discussed in Chapters II-IV. However, we shall now choose ϕ rather than V to denote this electric potential. This is partly for the sake of convention, but also to avoid confusion, for in Chapter XIII the symbol V will be needed to designate the velocity of electromagnetic waves in dielectric media.

There is one important deduction from (30). If a block of conducting metal is in the presence of a changing magnetic field, this shows that there is an electric field **E** induced in the body of the metal. Now $\mathbf{j} = \sigma\mathbf{E}$, so that there will be an induced current distribution. These are called eddy currents. In most cases (e.g. the core of a transformer or an electric motor) the eddy currents represent an unnecessary waste of energy, so the metal is usually laminated, with a thin film of insulation between the various layers. This serves to inhibit the undesirable eddy currents, though it does not completely remove them.

§ 98. Examples

1. A finite length of wire PQ is moved with constant velocity across a uniform magnetic field. Explain how you reconcile the existence of an e.m.f. $-dN/dt$ (equation 1) between P and Q with the fact that there is no complete circuit in which a current can flow.

2. A train is moving with velocity v. Show that if the two rails are insulated from the ground, there is a difference of potential between them equal to vlH, where l is the distance between the rails and H is the perpendicular component of the earth's magnetism.

3. A coil of area A, resistance R and induction L is rotated with constant angular velocity p about a vertical axis in the earth's magnetic field, whose horizontal component is H. Show that the current i satisfies the equation

$$L\frac{di}{dt} + Ri = AHp \sin pt.$$

By calculating the average rate of loss of energy show that the mean couple required to rotate the coil is $RA^2H^2p/2(R^2+L^2p^2)$.

4. A magnet of moment m can slide along the axis of a circular coil of wire of radius a, resistance R and negligible self-induction. Show that when the magnet is at a distance x from the coil the flux of B across the coil is $N = 2\pi ma^2/(a^2+x^2)^{3/2}$. Deduce that if the magnet is moved with constant velocity v the current in the coil is $i = 6\pi ma^2xv/R(a^2+x^2)^{5/2}$.

5. A small search-coil consists of n turns of wire each of area A, and the two ends are connected to a galvanometer, the total resistance being R. The coil is placed perpendicular to a field which it is desired to measure and is suddenly withdrawn. If the total charge that flows through the galvanometer is Q, show that the field is given by $B = QR/nA$. This is the principle of the **fluxmeter** by which we measure a field at any point.

6. A condenser is charged by means of a constant e.m.f. $\mathcal{E}$ the connecting wires having a resistance R. At $t = 0$ the e.m.f. is switched on. Show that at time t the charge on the condenser is $Q = C\mathcal{E}(1-e^{-t/CR})$.

7. Verify that the time constant τ in (10) is such that in time τ the current has grown to about 0·63 of its maximum value.

8. The two plates of a condenser C carry charges $\pm Q_0$, and at $t = 0$ they are connected together through a coil of resistance R and inductance L. Show that the charge Q at any subsequent time is given by $L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{Q}{C} = 0$. Solve this equation, given that $R^2 < 4L/C$.

9. Investigate the solutions of (14) for the free oscillations of a circuit in the cases $R^2 = 4L/C$ and $R^2 > 4L/C$. These are known as **critically damped** and **over-damped circuits.**

10. In the inductively coupled circuits of Fig. 67, $\mathcal{E}_2 = 0$ and $\mathcal{E}_1 = \mathcal{E}_0 \cos pt$. Show that there is a phase difference between i_1 and i_2, and that as $p \to \infty$ this phase difference tends to π.

11. In the inductively coupled circuits of Fig. 67, $\mathcal{E}_2 = 0$ and $\mathcal{E}_1 = \mathcal{E}_0 \cos pt$. Show that the current i_2 is given by the equation

$$(L_1L_2-M^2)\frac{d^2i_2}{dt^2} + (L_1R_2+L_2R_1)\frac{di_2}{dt} + R_1R_2i_2 = Mp\,\mathcal{E}_0 \sin pt.$$

Deduce that if the condition (24), $(L_1L_2 = M^2)$ is satisfied, then the amplitude of the oscillations of i_2 is $Mp\mathcal{E}_0/S$, where

$$S^2 = R_1{}^2R_2{}^2 + p^2(L_1R_2+L_2R_1)^2.$$

12. Show that in the forced oscillations of a single circuit (18) the current is out of phase with the e.m.f. Calculate the rate at which the e.m.f. $\mathcal{E}_0 \cos pt$ is working, and show that the mean value is $\dfrac{\mathcal{E}_0{}^2}{2Z} \cos \delta$, Z and δ being given by (19). Verify that this is the same as the mean value of Ri^2, so that conservation of energy is not violated. Deduce that if $R = 0$ there is no consumption of energy even though there may be a large current. Explain this.

13. A small magnet $\mathbf{m}$ at the origin is rotating about its centre with an angular velocity $\mathbf{p}$, the magnitude of $\mathbf{m}$ remaining constant. Show that $\dfrac{d\mathbf{m}}{dt} = \mathbf{p}\times\mathbf{m}$. Hence using the formula (6) in § 66

for $\mathbf{A}$ and (30) for $\mathbf{E}$ show that the motion of the magnet gives rise to an electric field $\mathbf{E}$, where

$$\mathbf{E} = \frac{1}{cr^3}\{(\mathbf{r}.\mathbf{m})\mathbf{p}-(\mathbf{r}.\mathbf{p})\mathbf{m}\}.$$

N.B.—We know that moving charges (i.e. currents) give rise to a magnetic field. This is an example to show that moving magnets give rise to an electrostatic field.

14. A small magnet $\mathbf{m}$ has a constant velocity $\mathbf{v}$. Show that at the moment when the vector distance from the magnet to a point P is $\mathbf{r}$ the electrostatic field at P due to the magnet's motion is

$$\mathbf{E} = \frac{\mathbf{m}\times\mathbf{v}}{cr^3} - \frac{3(\mathbf{v}.\mathbf{r})(\mathbf{m}\times\mathbf{r})}{cr^5}.$$

N.B.—This is only valid if v is much less than the velocity of light.

CHAPTER XII

ALTERNATING-CURRENT THEORY

§ 99. Introduction

WE are concerned in this chapter with some of the more technical applications of the fundamental laws of induction established in Chapter XI. Now in § 95 (*b*) of that chapter we considered the circuit of Fig. 66 in which an alternating e.m.f. $\mathcal{E} = \mathcal{E}_0 \cos pt$ was applied to a circuit containing a resistance R, a capacity C and an inductance L in series. We first obtained the differential equation (18) for i; then we saw that its solution involved two quite distinct parts. The one part (Complementary Function) represented oscillations which soon died away on account of damping; their frequency was quite independent of the frequency of the applied e.m.f. $\mathcal{E}$, and we described them as free oscillations of the circuit. But the other part (Particular Integral) represented oscillations with the same frequency as the applied e.m.f. and with constant amplitude; we called these forced oscillations. When the circuit is first completed with a switch, both parts of the total current are effective, but in a short time the transients have decayed to a negligible amplitude and only the forced oscillations remain. The distinction between these two types of oscillations is important, for in every electrical circuit of this kind both types are found. In our present chapter we shall be concerned solely with the forced oscillations, so that all transient effects will be neglected, and all oscillating quantities may be assumed to have the same frequency $p/2\pi$, though there may be certain phase differences between them. In such problems as these it is possible to simplify the working very considerably.

§ 100. Impedance and reactance

Let us first consider certain special cases of the general circuit shown in Fig. 68. In particular let us suppose that

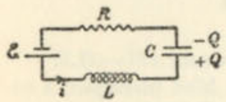


Fig. 68

$$\mathcal{E} = \mathcal{E}_0 \cos pt, \quad . \quad . \quad (1)$$

and that we connect, in turn, just one of the three quantities R, L and C to the e.m.f. $\mathcal{E}$. This may be achieved by supposing that the other two quantities have zero magnitude. The reader will have no difficulty in verifying that in these three cases the full solutions for the forced oscillations (equation (19) of § 95) reduce to

$$R \text{ alone}: \quad i = \frac{\mathcal{E}_0}{R} \cos pt, \quad . \quad . \quad . \quad . \quad (2)$$

$$L \text{ alone}: \quad i = \frac{\mathcal{E}_0}{pL} \cos\left(pt - \frac{\pi}{2}\right), \quad . \quad . \quad . \quad (3)$$

$$C \text{ alone}: \quad i = \frac{\mathcal{E}_0}{(pC)^{-1}} \cos\left(pt + \frac{\pi}{2}\right). \quad . \quad . \quad (4)$$

We may describe the results (2)-(4) by saying that in this particular circuit

(i) an inductance L behaves like an effective resistance pL, but the phase of the current lags $\pi/2$ behind the voltage,

(ii) a capacity C behaves like an effective resistance $1/pC$, but the phase of the current is $\pi/2$ in advance of the voltage,

(iii) a resistance R has no effect on the phase of the current.

We shall find a similar situation if we take R, L and C in pairs. The appropriate solutions now reduce to

$$R \text{ and } L \text{ together}: \quad i = \frac{\mathcal{E}_0}{Z} \cos(pt - \delta), \text{ where}$$

$$Z^2 = R^2 + p^2L^2, \ \tan\delta = pL/R, \quad . \quad . \quad . \quad (5)$$

$$R \text{ and } C \text{ together}: \quad i = \frac{\mathcal{E}_0}{Z} \cos(pt + \delta), \text{ where}$$

$$Z^2 = R^2 + 1/(pC)^2, \ \tan\delta = 1/pCR, \quad . \quad . \quad (6)$$

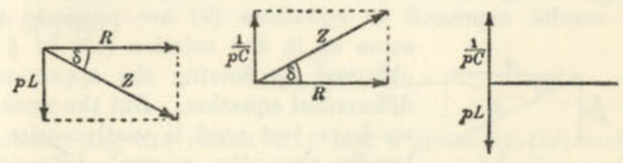
$$L \text{ and } C \text{ together}: \quad i = \frac{\mathcal{E}_0}{Z} \cos(pt - \pi/2) \text{ where}$$

$$Z = pL - 1/pC \quad . \quad . \quad . \quad . \quad (7)$$

We may describe the results (5)-(7) by saying that in each case there is an effective resistance Z, such that the maximum current is given by Ohm's law

$$i_{\max} = \mathcal{E}_0/Z, \quad . \quad . \quad . \quad . \quad (8)$$

and that there are phase differences which, except in the last case, are neither 0 nor $\pm\pi/2$. Evidently the actual phase differences in (5) and (6) represent a compromise between the effect of the pure resistance R which tends to leave the original phase unchanged, and that of L which tends to retard it, or of C which tends to advance it. Both Z and δ may be found graphically by drawing a right-angled triangle in which R is measured in one direction, whereas L and C are measured in a perpendicular direction. Z is then the magnitude of the vector sum of two components, and the magnitudes of these components are R, pL and $1/pC$. Fig. 69



(a) R and L together (b) R and C together (c) L and C together

Fig. 69

shows the three diagrams obtained in this way. Note, however, that in Fig. 69c, Z is the sum of the two vectors pL and $1/pC$. These are both perpendicular to the direction of R, but are opposed to one another. If $pL > 1/pC$ the

resultant points downward, but if $pL<1/pC$ it points up. Let us call the direction of R the **voltage line**; then if Z is above the voltage line the phase of the current is in advance of the e.m.f., and if Z lies below the voltage line the phase of the current is behind that of the e.m.f. In each case the magnitude of the phase difference is given by the angle between Z and the voltage line.

We call Z the **impedance** of the circuit. Thus Z is the magnitude of the vector sum of the **ohmic resistance** R and the **inductive reactance** pL, or the **capacitative reactance** $1/pC$, proper account being taken of the directions in which these two reactances must be measured. The quantity $1/Z$ is sometimes called the **admittance**.

Our argument has so far only been applied to the cases where either one or two of the three quantities are present. But it is equally valid when all three are present, as in Fig. 68. For, as Fig. 70 shows, the impedance Z of the circuit and the phase δ of the current are given by

$$Z^2 = R^2 + \left(pL - \frac{1}{pC}\right)^2, \quad \tan\delta = \left(pL - \frac{1}{pC}\right)\Big/ R,$$

with

$$i = \frac{\mathcal{E}_0}{Z}\cos(pt-\delta), \quad i_{\max} = \frac{\mathcal{E}_0}{Z}. \quad . \quad . \quad (9)$$

The results expressed in equations (9) are precisely the same as in the solution (19) of § 95 obtained by solving the appropriate differential equation. But the method we have just used is vastly easier to handle than the original differential equation, and, as we shall see shortly, it may be applied in many other more complicated examples.

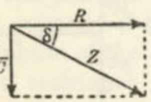
$pL - \frac{1}{pC}$ 

FIG. 70

101. Mathematical justification

The method developed above may be used in more general cases. In fact, it rests upon simple mathematical properties

of the Argand Diagram * and the complex numbers $a+jb$, where j denotes $\sqrt{-1}$. We use j rather than i for this quantity as i is already involved as the current. Instead of putting $\mathcal{E} = \mathcal{E}_0 \cos pt$, we shall find it convenient to use

$$\mathcal{E} = \mathcal{E}_0 e^{jpt}. \quad . \quad . \quad . \quad . \quad (10)$$

If we like we may interpret this as meaning that we are always to take the real part of all expressions involving e^{jpt}. But it will be quite good enough to work right through all our problems with functions of the type (10), without distinguishing between real and imaginary parts; then, when we have reached the end, we can take the real part.† This is a familiar device in Applied Mathematics. A similar type of expression to (10) will hold for the current; let us put

$$i = i_0 e^{jpt}, \quad . \quad . \quad . \quad . \quad (11)$$

but i_0 may not necessarily be purely real, as we shall see in a moment. Indeed, our primary object is to discover the modulus and argument of i_0. The type of substitution represented in (10) and (11) is possible because we have decided to consider only the forced oscillations in which all fluctuating quantities have the same frequency $p/2\pi$.

Let us consider first the circuit containing $\mathcal{E}$, L and R, all in series. Ohm's law for this circuit gives (equation (8) of § 95):

$$\mathcal{E} - L\frac{di}{dt} = Ri \quad . \quad . \quad . \quad . \quad (12)$$

Substituting the values of $\mathcal{E}$ and i given by (10) and (11) we see that (12) is satisfied if $\mathcal{E}_0 - Ljpi_0 = Ri_0$, that is, if

$$i_0 = \frac{\mathcal{E}_0}{R+jpL}. \quad . \quad . \quad . \quad . \quad (13)$$

* E.g. see Phillips, *Complex Variable*, Oliver and Boyd Ltd., Chapter I.

† E.g. see Coulson, *Waves*, Oliver and Boyd Ltd., 1949, Chapters I and II.

Thus i_0 is a complex quantity. To separate real and imaginary parts let us put

$$R+jpL = Ze^{j\delta} \quad . \quad . \quad . \quad . \quad (14)$$

It will be recognised at once that Z and δ thus defined are in fact precisely the same as in (5), so that Z is the impedance and δ is the phase lag of current behind e.m.f. Combining (11), (13) and (14)

$$\begin{aligned} i &= i_0 e^{jpt} \\ &= \frac{\mathcal{E}_0}{Z} e^{j(pt-\delta)} \quad . \quad . \quad . \quad (15) \end{aligned}$$

Taking the real part of this equation we reproduce in detail the formulæ in (5). The complex number which we introduced in (14) and which we have seen gives us both the impedance and the phase is often written $\mathbf{Z}$, and is called the **vector impedance**. When drawn on the Argand diagram its modulus is the impedance Z and its argument is the phase δ. In fact

$$\mathbf{Z} = Ze^{j\delta} \quad . \quad . \quad . \quad . \quad (16)$$

$\mathbf{Z}$ may be regarded as composed of the ohmic resistance (or real part) R, and the inductive reactance (or complex part) jpL. The relation of all this to the Argand diagram is soon recognised in Fig. 69.

The method just described is quite general in its application. Thus, suppose that we have the circuit of Fig. 68 containing L, R and C all in series, the fundamental equations are (11) and (12) of § 95:

$$\mathcal{E} - L\frac{di}{dt} - \frac{Q}{C} = Ri \, , \, i = \frac{dQ}{dt} \quad . \quad . \quad (17)$$

To solve these, we put

$$\mathcal{E} = \mathcal{E}_0 e^{jpt} \, , \, i = i_0 e^{jpt} \, , \, Q = Q_0 e^{jpt} \quad . \quad . \quad (18)$$

Then with a little reduction

$$i_0 = jpQ_0, \quad . \quad . \quad . \quad . \quad (19)$$

$$\left\{R + jpL + \frac{1}{jpC}\right\} i_0 = \mathcal{E}_0 \quad . \quad . \quad . \quad (20)$$

The vector impedance is

$$\mathbf{Z} = R + jpL + \frac{1}{jpC}, \quad . \quad . \quad . \quad (21)$$

and

$$i = \frac{\mathcal{E}}{\mathbf{Z}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

It is easily recognised that these equations lead to precisely the same results as in (9), which is known to be the correct solution.

We could, however, apply the same analysis to any other circuit, or part of a circuit. For as we saw in Chapter V the current distribution in electrical networks is governed by Kirchhoff's laws; these state that no current piles up at any junction of wires, and that the potential drop round any closed contour is zero, making allowance for batteries that may be present. With alternating currents of frequency $p/2\pi$, all these conditions will resemble those of (17), with perhaps more terms according to the complexity of the contour, and continuity equations for the current. Now all currents and charges will vary with the time according to the law e^{jpt}, so that we may always replace quantities such as $L\frac{di}{dt}$ by $jpLi$, and $\frac{Q}{C}$ by $\frac{1}{jpC}i$. Thus each inductance may be treated as a complex resistance jpL and each condenser as a complex resistance $\frac{1}{jpC}$; after this we may proceed to solve the equations for the currents by using Ohm's law, just as in (22). The final current, or currents, will be given by complex expressions, whose moduli are the maximum values and whose arguments determine the phases.

We have just seen that a resistance R and an inductance L in series may be regarded as a single vector impedance $\mathbf{Z}$, where $\mathbf{Z} = R + jpL$. In a similar way we may treat other combinations, so that a complete network may be broken up into a set of impedances $\mathbf{Z}_1, \mathbf{Z}_2, \ldots$ This is often easier than treating each part separately, for it is easy to show that impedances in series are compounded together according to the law

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 + \ldots \quad . \quad . \quad . \quad (23)$$

but if they are in parallel

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \ldots \quad . \quad . \quad . \quad (24)$$

It is important to remember that (23) and (24) require us to use the vector impedance $\mathbf{Z}$ and not its magnitude Z. We leave the proofs of (23) and (24) as an exercise for the student; examples of their use are given below.

§ 102. Bridge circuits

By replacing the resistances of a Wheatstone's bridge (§ 41) by condensers or inductances, or both, we are able to obtain various modifications, some of which are important as they allow us to compare inductances and capacities and resistances.

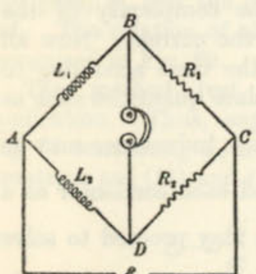


Fig. 71

Thus, consider first the **De Sauty bridge** shown in Fig. 71. AB and AD are inductances L_1 and L_2, but BC and CD are resistances R_1 and R_2. The alternating supply $\mathcal{E} = \mathcal{E}_0 e^{jpt}$ is applied between A and C, and the resistance R_2 is adjusted so that no current flows across BD; this is determined by a pair of headphones, shown in the diagram.

We may treat this circuit just like the Wheatstone's bridge in § 41, taking the four "resistances" as jpL_1, jpL_2, R_1 and R_2. The condition for balance now becomes

$$\frac{jpL_1}{jpL_2} = \frac{R_1}{R_2}, \text{ so } L_1 : L_2 = R_1 : R_2 \quad . \quad . \quad (25)$$

In this way we compare the inductances L_1 and L_2.

If we had replaced the inductances L_1 and L_2 by two condensers C_1 and C_2, the condition for balance would have been

$$\frac{jpC_2}{jpC_1} = \frac{R_1}{R_2}, \text{ so that } C_1 : C_2 = R_2 : R_1. \quad . \quad (26)$$

Both (25) and (26) are independent of the frequency p. This does not often happen, but when it does, it means that we need not use a purely harmonic e.m.f. Any type of e.m.f. will serve, provided that it is not a constant one; this is because we can split up the variations of e.m.f. into harmonic components by Fourier's theorem, and since (25) and (26) are true for each component of the e.m.f. they are true for the complete e.m.f. A satisfactory method in such cases is to use a battery with a make-and-break switch.

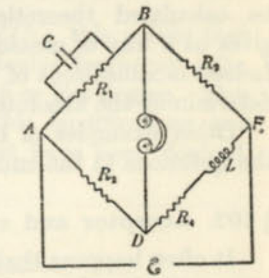


Fig. 72

Next consider the **Maxwell bridge** shown in Fig. 72. The four arms of the bridge contain ohmic resistances $R_1 \ldots R_4$, but R_1 is in parallel with a condenser C and R_4 is in series with an inductance L. Once again the applied e.m.f. is $\mathcal{E}_0 e^{jpt}$, connected between A and F, and adjustments are made till no current flows through the earphones between B and D. If $\mathbf{Z}_1 \ldots \mathbf{Z}_4$ are the vector impedances, the condition of balance is

$$\mathbf{Z}_1\mathbf{Z}_4 = \mathbf{Z}_2\mathbf{Z}_3. \quad . \quad . \quad . \quad (27)$$

Now $\mathbf{Z}_2 = R_2$, $\mathbf{Z}_3 = R_3$, $\mathbf{Z}_4 = R_4+jpL$, and

$$\frac{1}{\mathbf{Z}_1} = \frac{1}{R_1} + \frac{1}{(1/jpC)}.$$

Substituting in (27) we find, after a little reduction

$$R_1(R_4+jpL) = R_2R_3(1+jpCR_1).$$

Equating the real and imaginary parts:

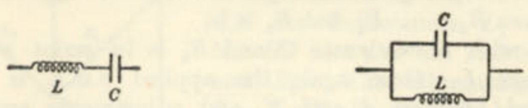
$$R_1R_4 = R_2R_3, \text{ and } L = CR_2R_3 \quad . \quad . \quad (28)$$

There are thus two distinct conditions; this often occurs in problems of this nature; it arises because the currents are specified not merely by their maximum values but also by their phases. In our present problem the conditions of balance are independent of p, just as in the De Sauty bridge. The final relation in (28) is very useful because if we are able to calculate any two of the quantities capacity, resistance and self-inductance, it gives us a method of measuring the other. We have seen in earlier chapters that L and C may be calculated theoretically; in principle, therefore, this gives us a way of measuring resistances on an absolute scale. Indeed modifications of this circuit were used by Maxwell in determining the absolute ohm.

Other examples of bridges of this kind will be found in the questions at the end of the chapter.

§ 103. Acceptor and rejector circuits

It often happens that we wish either to allow or to exclude one particular frequency of electric oscillations. For this



(a) acceptor (b) rejector

Fig. 73

purpose we use an acceptor or rejector circuit, as shown in Fig. 73.

Consider first the circuit of Fig. 73*a*. The effective impedance is

$$\mathbf{Z} = jpL + \frac{1}{jpC} = j\left(pL - \frac{1}{pC}\right).$$

So if $p^2 = 1/(LC)$, $\mathbf{Z}$ vanishes and the system allows free transmission for the particular frequency $p/2\pi$. All other frequencies give a non-zero value for $\mathbf{Z}$, and are therefore cut down in greater or less degree. This is called an acceptor circuit.

Next consider the circuit of Fig. 73*b*. The effective impedance is given by

$$\frac{1}{\mathbf{Z}} = \frac{1}{jpL} + \frac{1}{(1/jpC)}.$$

Thus

$$\mathbf{Z} = \frac{jpL}{1-p^2LC}.$$

Consequently $\mathbf{Z}$ is infinite for $p^2 = 1/(LC)$; this means that the circuit completely rejects the particular frequency $p/2\pi$, admitting other frequencies to a greater or less degree. This is called a rejector circuit. Considerable modifications and variations of these two types of circuit have been devised, but we shall not discuss them here.

§ 104. Power factor

Consider again the series-resonance circuit shown in Fig. 66 (§ 95). The applied e.m.f. is, in real form,

$$\mathcal{E} = \mathcal{E}_0 \cos pt,$$

and the current, also in real form, is

$$i = \frac{\mathcal{E}_0}{Z} \cos (pt-\delta). \quad . \quad . \quad (29)$$

Now the rate at which the source of supply is providing

energy is $\mathcal{E}i$. This is a fluctuating quantity whose mean value is $\frac{\mathcal{E}_0^2}{2Z}\cos\delta$. This may be described as the true power consumption; it is the quantity that would be measured by a watt-meter.

It is, however, quite possible to measure the current and voltage separately with an a.c. ammeter and voltmeter respectively. Instruments of this sort do not measure the mean value, which is zero, but the root mean square value (r.m.s.). Thus the current recorded would be the square root of the average of i^2, which is soon shown to be $\mathcal{E}_0/\sqrt{2}Z$, on account of the fact that the mean value of $\cos^2 pt$ is $\frac{1}{2}$. Similarly the r.m.s. voltage is $\mathcal{E}_0/\sqrt{2}$; the product of the r.m.s. voltage and r.m.s. current may be called the apparent power. It differs from the true power by the factor $\cos\delta$, which we call the power factor. Evidently the power factor lies between 0 and 1, taking the value 0 when the current is right out of phase with the e.m.f. and the value 1 when the two are completely in phase. In simple terms

true power = r.m.s. voltage × r.m.s. current × power factor.

A large phase lag between current and voltage gives a low power factor; this is wasteful in distribution since the same true power requires a larger r.m.s. current and hence larger losses through Joule heat in the wires. For this reason electric power companies instal devices to correct a low power factor and bring it as high as 0·9 or higher. When designing these devices we have to remember that condensers advance the phase of the current and inductances retard it.

§ 105. Examples

1. An alternating current $i_2 \cos pt$ is superimposed upon a direct current i_1. Show that the r.m.s. current is $\sqrt{(i_1^2+\frac{1}{2}i_2^2)}$.

2. An e.m.f. represented by $\mathcal{E}_0+\mathcal{E}_1\cos pt$, where $\mathcal{E}_0$ and $\mathcal{E}_1$ are real constants, acts in a circuit whose vector impedance is $\mathbf{Z} = Ze^{j\delta}$. Show that the mean power expended in the circuit is $\{\mathcal{E}_0^2 \sec\delta+\frac{1}{2}\mathcal{E}_1^2\cos\delta\}/Z$.

3. Prove equations (23) and (24) for impedances in series and parallel.

4. Show that the impedance of the system in Fig. 74 (*a*) is $\alpha+jp\beta$, where $\alpha = R/(1+p^2C^2R^2)$, $\beta = L-(CR^2)/(1+p^2C^2R^2)$. Hence show that if pCR is very much less than 1, it is possible to choose R and C so that there is zero resultant reactance. This device, which may be thought of as cancelling the inductance L, is useful in long-distance cables, where the presence of an inductance is undesirable.

5. Show that in the transformer of Fig. 74 (*b*) the mutual inductance M is equivalent to an impedance $jpMi_2/i_1$ in circuit 1 and $jpMi_1/i_2$ in circuit 2.

6. Show that the forced oscillations in the transformer circuit Fig. 74 (*c*) are given by

$$\mathcal{E} = jpL_1i_1+jpMi_2\,,\ 0 = Ri_2+jpL_2i_2+jpMi_1.$$

Deduce that the peak value of the current in the secondary coil is $M\mathcal{E}_0/\sqrt{\{(L_1L_2-M^2)^2p^2+L_1^2R^2\}}$.

7. Show that if the coefficient of coupling $M/\sqrt{(L_1L_2)}$ in the last question is unity, then the power factor is $pL_2/\sqrt{(R^2+p^2L_2^2)}$.

8. Show that the natural frequencies of the circuit in Fig. 74 (*d*) are $p/2\pi$, where $C_1C_2(L_1L_2-M^2)p^4-(L_1C_1+L_2C_2)p^2+1 = 0$.

9. Show that the power factor for the circuit shown in Fig. 74 (*e*) is $pL/\sqrt{\{p^2L^2+R^2(p^2LC-1)^2\}}$.

10. Figure 74 (*f*) shows two circuits with a common capacity C. This is called **direct capacity coupling**. Show that the periods of free oscillation are $2\pi/p$, where

$$\{CC_1L_1p^2-(C+C_1)\}\,\{CC_2L_2p^2-(C+C_2)\} = C_1C_2.$$

11. In the circuit of Fig. 74 (*g*) the condenser is adjusted so that no current flows through the telephone. Prove that $p^2MC = 1$. This may be regarded as a rejector circuit; it is called the **Campbell Frequency Sifter**. Notice that it provides a measure of M, even though self-inductances are present in both halves of the circuit.

12. Show that the conditions for balance in the **Anderson bridge** of Fig. 74 (*h*) are

$$R_1R_4 = R_2R_3,\ L = CR_3(R_1R_2+R_1R_5+R_2R_5)/R_1.$$

13. In the Bridge circuit of Fig. 74 (*i*) there is a mutual

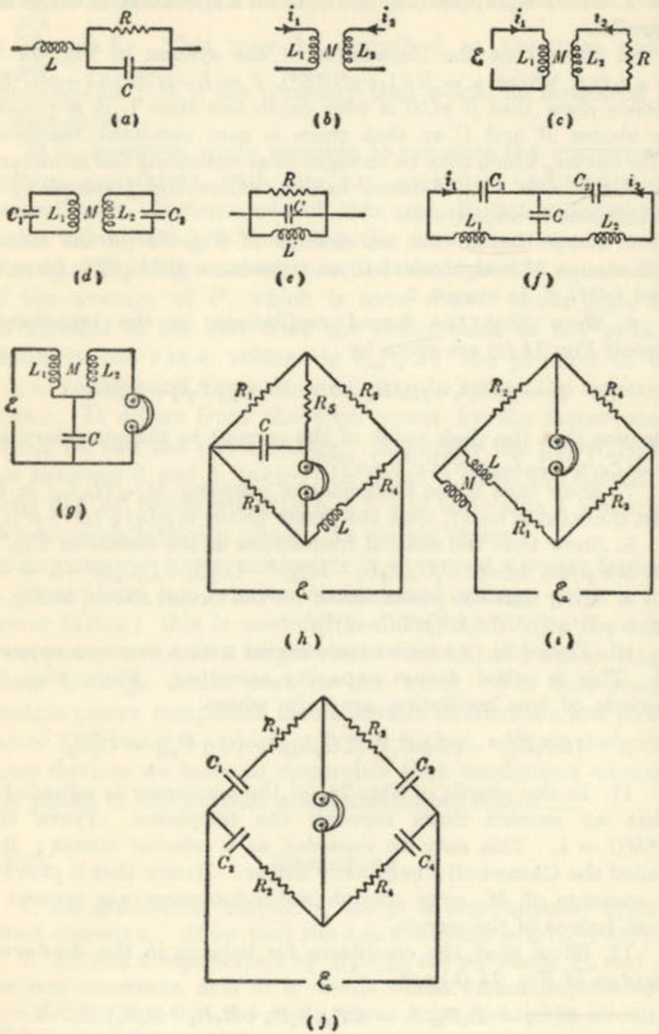


FIG. 74

inductance M between the incoming wire and the arm containing the resistance R_1. Show that the conditions of balance are

$$R_1R_4 = R_2R_3,\ LR_4 = -M(R_3+R_4).$$

14. Show that the conditions of balance in the circuit of Fig. 74 (j) are

$$R_1R_4 - R_2R_3 = \frac{C_2C_3 - C_1C_4}{p^2C_1C_2C_3C_4},\ \frac{R_4}{C_1} + \frac{R_1}{C_4} = \frac{R_2}{C_3} + \frac{R_3}{C_2}.$$

In all the circuit diagrams above, $\mathcal{E}$ represents an e.m.f. $\mathcal{E}_0e^{jpt}$, and L, M, C and R represent self-inductance, mutual inductance, capacity and resistance respectively.

CHAPTER XIII

MAXWELL'S EQUATIONS

§ 106. Displacement current

THE fundamental equations of the electromagnetic field are not yet completely established, for one of our previous equations requires an additional term. It was Maxwell who detected the omission and thus discovered a new and unsuspected importance in the study of electromagnetism.

The equation that requires modification is the one established in Chapter VI which connects the current vector $\mathbf{j}$ and the associated magnetic field $\mathbf{H}$ in the form

$$\operatorname{curl} \mathbf{H} = 4\pi\mathbf{j} \quad . \quad . \quad . \quad (1)$$

Now $\operatorname{div} \operatorname{curl} \equiv 0$, so that (1) automatically implies that $\operatorname{div} \mathbf{j} = 0$. But the equation of continuity of charge (§ 33, equation (6)) is

$$\operatorname{div} \mathbf{j} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{1}{4\pi} \operatorname{div} \mathbf{D}\right) \quad . \quad . \quad (2)$$

Thus in general $\operatorname{div} \mathbf{j}$ is not zero; but it is zero for the case of steady flow discussed in Chapter VI. Equation (1) does not therefore hold for non-steady currents. However we may write the equation of continuity (2) in the form

$$\operatorname{div} \mathbf{J} = 0,$$

where

$$\mathbf{J} = \mathbf{j} + \frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t} \quad . \quad . \quad . \quad (3)$$

This suggests very strongly that we ought to replace $\mathbf{j}$ in (1) by $\mathbf{J}$. For $\mathbf{J}$ is a vector whose divergence is always zero and

which reduces to $\mathbf{j}$ in steady cases. We call $\mathbf{J}$ the **total current**, $\mathbf{j}$ the **conduction current**, and $\frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t}$ the **displacement current**. Since $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ (§ 26), the total current may be written

$$\mathbf{J} = \mathbf{j} + \frac{\partial \mathbf{P}}{\partial t} + \frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t}. \quad . \quad . \quad . \quad (4)$$

We can give a simple explanation of part of the displacement current. For when we polarise a dielectric we separate positive and negative charges. While this separation is taking place charges are moving and a current $\frac{\partial \mathbf{P}}{\partial t}$, called the **polarisation current**, must flow. This justifies the second term in (4). The third term, introduced by Maxwell, is sometimes called the **aether displacement current**, or displacement current *in vacuo*. The total current is thus a sum of three separate terms, and the revised form of (1) is

$$\operatorname{curl} \mathbf{H} = 4\pi\mathbf{J} = 4\pi\mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}. \quad . \quad . \quad (5)$$

If we measure $\mathbf{H}$ and $\mathbf{j}$ in e.m.u., and $\mathbf{D}$ in e.s.u., this becomes

$$\operatorname{curl} \mathbf{H} = 4\pi\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \quad . \quad . \quad . \quad (6)$$

The step from (1) to (6) has been shown to be reasonable and consistent. It is, however, possible to verify its accuracy experimentally by applying an alternating e.m.f. to the plates of a condenser between which there is a uniform dielectric. In the dielectric there is no conduction current $\mathbf{j}$, but according to (6) there should be a displacement current $\frac{1}{4\pi c}\frac{\partial \mathbf{D}}{\partial t}$. It is not difficult to establish the presence of this current by its magnetic field. In this way, and in other ways to be seen later, equation (6) is fully justified. Indeed the whole of the electromagnetic theory of light depends upon the inclusion of the extra terms shown in (4).

In previous chapters we have neglected the displacement current. This is certainly accurate for steady conditions (in which $\partial D/\partial t = 0$), and it is sufficiently accurate if the rate of change of **D** is not large. Such a condition is known as a **quasi-steady state.** All the equations and deductions in our discussion of induction in Chapter XI are thus dependent upon the assumption of a quasi-steady state. A study of numerical values shows that we may treat all ordinary electrical engineering problems as quasi-steady problems: it is only when we reach very high frequencies, such as those of light, that the displacement current predominates over the conduction current.

§ 107. Maxwell's equations

We are now in a position to summarise the fundamental equations of the electromagnetic field. They are:

$$\text{(i) } \operatorname{div} \mathbf{D} = 4\pi\rho,$$
$$\text{(ii) } \operatorname{div} \mathbf{B} = 0,$$
$$\text{(iii) } \operatorname{curl} \mathbf{H} = 4\pi\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t},$$
$$\text{(iv) } \operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}.$$

Here, and elsewhere in this chapter we use mixed, or Gaussian units, i.e. **D**, **E**, ρ are measured in e.s.u., and **B**, **H**, **j** in e.m.u. Equations (i)-(iv) are known briefly as **Maxwell's Equations.** But they need to be supplemented by the **constitutive relations**:

$$\text{(v) } \mathbf{D} = K\mathbf{E}, \quad \text{(vi) } \mathbf{B} = \mu\mathbf{H}, \quad \text{(vii) } \mathbf{j} = \sigma\mathbf{E}.^*$$

To this list we may add the equations which define the potentials **A** and ϕ. We have already seen that these are:

$$\text{(viii) } \mathbf{B} = \operatorname{curl} \mathbf{A},$$
$$\text{(ix) } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad} \phi.$$

* Depending on whether **j**, σ and **E** are in the same units, a factor c may be required. See the last line in § 119.

Equation (viii) does not completely define **A**. In the steady conditions of § 58, equations (12) and (13), we added the extra condition $\operatorname{div} \mathbf{A} = 0$. But when dealing with fluctuating currents it is more convenient to generalise this in the form

$$\text{(x) } \operatorname{div} \mathbf{A} + \frac{K\mu}{c}\frac{\partial \phi}{\partial t} = 0.$$

In steady and quasi-steady states this new definition is the same as the old, and none of our earlier work is affected by the alteration. But the new form greatly simplifies the actual calculation of **A** and ϕ in non-steady states.

§ 108. Decay of free charge

Several important deductions can be made from (i)-(x). Thus, using (5) of p. 225 and (vii) of p. 226, we may write

$$\operatorname{curl} \mathbf{H} = 4\pi\sigma\mathbf{E} + \frac{\partial \mathbf{D}}{\partial t}.$$

Let us suppose that σ and K are constant. Then taking the divergence of each side and using (i) we have

$$0 = \frac{4\pi\sigma}{K}\rho + \frac{\partial \rho}{\partial t}.$$

Thus, on integration:

$$\rho = \rho_0 e^{-t/\theta}, \text{ where } \theta = K/4\pi\sigma. \quad . \quad . \quad (7)$$

θ is called the **time of relaxation.** It follows from (7) that any original distribution of charge decays exponentially at a rate quite independent of any other electromagnetic disturbances that may be taking place simultaneously, and it justifies us in putting $\rho = 0$ in most of our problems. For metals θ is too small to measure with any accuracy (e.g. for copper θ is of the order of 10^{-18} seconds), but for dielectrics such as water the experimental value agrees excellently with (7). This theorem does not, of course, apply to charges at the surface of a conductor.

§ 109. Electric waves

Perhaps the most important deduction from Maxwell's equations is that they predict electric waves. Consider a non-conducting medium such as glass, for which we may put $\sigma = 0$; let us also suppose that K and μ are constants. Then operating with curl on both sides of (iii) and using the fact that curl curl $\equiv$ grad div$-\nabla^2$, we get:

$$\text{grad div } \mathbf{H} - \nabla^2\mathbf{H} = \frac{K}{c}\,\text{curl}\,\frac{\partial \mathbf{E}}{\partial t} = \frac{K}{c}\frac{\partial}{\partial t}(\text{curl}\,\mathbf{E}).$$

Substituting for div $\mathbf{H}$ and curl $\mathbf{E}$, we obtain the equation

$$\nabla^2\mathbf{H} = \frac{K\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}. \quad . \quad . \quad . \quad (8)$$

Eliminating $\mathbf{H}$ instead of $\mathbf{E}$ we find the same equation for $\mathbf{E}$:—

$$\nabla^2\mathbf{E} = \frac{K\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}. \quad . \quad . \quad . \quad (9)$$

These are the standard equations of wave motion.* They show that electric waves are transmitted with velocity $c/\sqrt{(K\mu)}$. In free space, where $K = \mu = 1$, this velocity is just c. Now c, which first appeared in Chapter VI as the ratio of the e.m.u. and e.s.u. of current, may be determined experimentally: its value in c.g.s. units is $2{\cdot}998 \times 10^{10}$. But it is found by experiment that the velocity of light in free space has this same value. We are thus led to the conviction that light waves are electromagnetic in nature, a view that has subsequently received complete verification. X-rays, ultra-violet rays, infra-red rays and wireless waves are also electromagnetic and differ only in the order of magnitude of their wavelengths. It can be shown, though we shall not do so here, that they are all transverse waves.

In non-conducting dielectric media, like glass, K is not

* See Coulson, *Waves*, Oliver and Boyd Ltd., 1949, Chapters I and VII, where a fuller account of electromagnetic waves is given.

equal to unity; also μ depends on the frequency of the waves, but for light waves in the visible region we may put $\mu = 1$. The velocity of light is therefore $c/\sqrt{K}$. Now in a medium whose refractive index is n, it is known experimentally that the velocity of light is c/n. Hence if our original assumptions are valid, $K = n^2$. This result, which is known as **Maxwell's relation**, is approximately satisfied by many substances, but it fails because it does not take sufficiently detailed account of the atomic structure of the dielectric. It applies better for long waves (low frequency) than for short waves (high frequency).

The student may easily verify that a particular solution of (8) and (9) is

$$\begin{aligned} &E_x = a \cos p(t - z/V)\;,\quad E_y = E_z = 0, \\ &H_y = \sqrt{(K/\mu)}\,a \cos p(t - z/V)\;,\quad H_x = H_z = 0 \quad . \quad (10) \end{aligned}$$

in which $V = c/\sqrt{(K\mu)}$ and a and p are constants. This evidently represents a plane wave of frequency $p/2\pi$ travelling in the z-direction with velocity V. The vectors $\mathbf{E}$, $\mathbf{H}$ and the direction of propagation, form a right-handed set of axes, showing that such waves are transverse and not longitudinal.

§ 110. Conducting media

In our previous discussion we dealt only with non-conducting media, where $\sigma = 0$. If we retain σ, the standard equations (8) and (9) are modified by the presence of an extra term. Thus, supposing always in accordance with § 108, that $\rho = 0$,

$$\nabla^2\mathbf{E} = \frac{K\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \frac{4\pi\sigma\mu}{c}\frac{\partial\mathbf{E}}{\partial t}. \quad . \quad . \quad (11)$$

There is a similar equation for $\mathbf{H}$. This is the well-known **equation of telegraphy**, from which we can infer that the additional term corresponds to a dissipation of energy. If we are dealing with waves whose frequency is $p/2\pi$, $\mathbf{E}$ will be proportional to e^{ipt} (or $\cos pt$, or $\sin pt$) and we may

replace $\partial/\partial t$ by ip. The ratio of the absolute magnitudes of the two terms on the right-hand side of (11) is therefore $Kp/4\pi\sigma c$. If p is very large, as in light waves in a non-conducting dielectric, only the first term (displacement term) matters; but if p is small, as in long waves in a good conductor, the second term (conduction term) predominates. If we confine ourselves to the latter problem we may omit the term $\frac{\partial^2\mathbf{E}}{\partial t^2}$ in (11). Putting $\mathbf{j} = \sigma\mathbf{E}$, we obtain the differential equation of current flow:

$$\nabla^2\mathbf{j} = \frac{4\pi\sigma\mu}{c}\frac{\partial\mathbf{j}}{\partial t} \quad . \quad . \quad . \quad (12)$$

This is similar to the equation governing the flow of heat, with current $\mathbf{j}$ replacing the temperature. This analogy enables us to predict several important results, of which we may briefly refer to one.

If the temperature at the plane face of a semi-infinite slab is fluctuating with given frequency, then as we proceed inside the slab there will be fluctuations of the same frequency, but of decreasing amplitude and increasing phase lag. Correspondingly it follows that if an alternating current flows in a conductor, the amplitude of the current decreases as we proceed into the conductor, and there is also a phase lag. Rapidly alternating current, therefore, tends to confine itself to the surface. This phenomenon is known as the **skin effect**, and has an important effect upon the resistance of wires in high frequency circuits.

§ 111. Poynting vector

We must next discuss the energy of an electromagnetic field. Since the electrostatic energy (§ 28) is $\int\frac{(\mathbf{E}.\mathbf{D})}{8\pi}\,dv$, and by analogy (§ 64) the magnetic energy is $\int\frac{(\mathbf{H}.\mathbf{B})}{8\pi}\,dv$, we should expect the total energy to be the sum of the two. This may, however, be proved by considering the rate of change of energy in a given volume; for we have seen that all electromagnetic effects may be attributed to charges, at rest or in motion; and we know that a moving charge experiences a force, so that as it moves, work is continually being done on it. In what follows we shall calculate the rate at which the energy changes due to these forces exerted on each moving charge.

The force on a unit charge (§ 62) is $\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}$. Thus the rate at which work is done on unit charge is

$$\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{B})\right].\mathbf{v} = (\mathbf{E}.\mathbf{v}).$$

Hence the total rate at which the forces of the field are doing work in the given volume is

$$\int(\mathbf{E}.\mathbf{v})\rho\,dv = c\int(\mathbf{E}.\mathbf{j})\,dv, \text{ using mixed units.}$$

Substituting for $\mathbf{j}$ from Maxwell's Equations (§ 107, (iii)), this becomes

$$\frac{c}{4\pi}\int\mathbf{E}.\operatorname{curl}\mathbf{H}\,dv - \frac{1}{4\pi}\int\mathbf{E}.\frac{\partial\mathbf{D}}{\partial t}\,dv,$$

or, since $\quad \operatorname{div}(\mathbf{H}\times\mathbf{E}) = \mathbf{E}.\operatorname{curl}\mathbf{H} - \mathbf{H}.\operatorname{curl}\mathbf{E},$

$$\frac{c}{4\pi}\int\operatorname{div}(\mathbf{H}\times\mathbf{E})\,dv + \frac{c}{4\pi}\int\mathbf{H}.\operatorname{curl}\mathbf{E}\,dv - \frac{1}{4\pi}\int\mathbf{E}.\frac{\partial\mathbf{D}}{\partial t}\,dv.$$

The first integral may be transformed into a surface integral over the boundary of the given volume. Further, using (iv) in § 107 for curl $\mathbf{E}$, we obtain

$$\frac{c}{4\pi}\int(\mathbf{H}\times\mathbf{E}).d\mathbf{S} - \frac{1}{4\pi}\int\left\{\mathbf{H}.\frac{\partial\mathbf{B}}{\partial t} + \mathbf{E}.\frac{\partial\mathbf{D}}{\partial t}\right\}dv$$

as the rate at which the field is doing work. If μ and K are constants with respect to t, this may be written

$$-\frac{\partial}{\partial t}\int\left\{\frac{\mathbf{B}.\mathbf{H}}{8\pi}+\frac{\mathbf{D}.\mathbf{E}}{8\pi}\right\}dv=$$

$$\int\frac{c}{4\pi}(\mathbf{E}\times\mathbf{H}).d\mathbf{S}+\text{rate at which field does work.} \quad (13)$$

It is now quite clear that we must take the electromagnetic energy to be

$$\frac{1}{8\pi}\{\mathbf{B}.\mathbf{H}+\mathbf{D}.\mathbf{E}\} \quad . \quad . \quad . \quad (14)$$

per unit volume; energy conservation is then assured by (13) provided that we suppose the surface integral in (13) to represent a rate of flow, or radiation, of energy outward across the boundary of the volume. The vector

$$\frac{c}{4\pi}(\mathbf{E}\times\mathbf{H}), \quad . \quad . \quad . \quad . \quad (15)$$

which gives the rate at which energy is radiated across unit area, is known as the **Poynting vector.** In the form in which we have proved it in (13), only the integral of the Poynting vector over a closed surface has physical significance. In most cases, however, e.g. radiation from aerials or light waves, in which the field is rapidly alternating about a mean value zero, the vector $\frac{c}{4\pi}(\mathbf{E}\times\mathbf{H})$ does represent the rate of flow across any isolated unit area.

Thus consider the plane wave defined by (10). It is easy to see that $\frac{c}{4\pi}(\mathbf{E}\times\mathbf{H})$ is directed along the positive z-axis and its magnitude is

$$\sqrt{\frac{K}{\mu}}.\frac{ca^2}{4\pi}.\cos^2 p\left(t-\frac{z}{V}\right).$$

This is a rapidly fluctuating quantity whose mean value is $\frac{1}{8\pi}\sqrt{\frac{K}{\mu}}.ca^2$, and this is the rate at which energy crosses unit area in the xy plane. Now, from (14) the energy density is

$$\frac{Ka^2}{4\pi}\cos^2 p\left(t-\frac{z}{V}\right),$$

with a mean value $Ka^2/8\pi$. If we may suppose that all the energy has a definite velocity of flow, this velocity must be the ratio $\frac{\text{rate of flow}}{\text{density}}$, i.e. $c/\sqrt{(K\mu)}$, or V. With such a wave, energy flows at the same speed as the wave itself. With a combination of two or more waves, however, for which V usually has different values depending on the frequency $p/2\pi$, this simple result is not true.*

§ 112. The potentials

Our discussion of electric waves in § 109 can equally well be given in terms of the potentials $\mathbf{A}$ and ϕ. We leave it as an exercise for the student to show that if $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{D}$ are eliminated from Maxwell's equations (i)-(x), we obtain the equations

$$\nabla^2\mathbf{A}=\frac{K\mu}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-4\pi\mu\mathbf{j}, \quad . \quad . \quad . \quad (16)$$

$$\nabla^2\phi=\frac{K\mu}{c^2}\frac{\partial^2\phi}{\partial t^2}-\frac{4\pi\rho}{K}. \quad . \quad . \quad . \quad (17)$$

In uncharged media, where $\rho=0$, (17) reduces to the standard equation of wave motion; so also does (16) in non-conducting media where $\mathbf{j}=0$. In either case this demonstrates the possibility of electric waves with velocity $c/\sqrt{(K\mu)}$.

* See e.g. Coulson, *Waves*, Oliver and Boyd Ltd., Chapter VIII.

§ 113. Retarded potentials

Let us consider for a moment the particular case in which $K = \mu = 1$. (16) and (17) become

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - 4\pi\rho, \quad . \quad . \quad . \quad (18)$$

$$\nabla^2\mathbf{A} = \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - 4\pi\mathbf{j}. \quad . \quad . \quad . \quad (19)$$

The solutions of these two equations may be obtained by an extension of the method developed in § 13.* The result for ϕ is

$$\phi = \int \frac{[\rho]_{t-r/c}}{r}\,dv \quad . \quad . \quad . \quad (20)$$

Similarly taking each component of **A** separately and then combining them

$$\mathbf{A} = \int \frac{[\mathbf{j}]_{t-r/c}}{r}\,dv. \quad . \quad . \quad . \quad (21)$$

The notation $[\rho]_{t-r/c}$ means that when integrating (20) to get the value of ϕ at a particular time t and a particular point P, each element $\rho\,dv$ contributes an amount $\rho\,dv/r$, but ρ is measured not at the time t but at a previous time $t-r/c$. Now r/c is precisely the time that an electromagnetic wave would take to go from dv to the point P where ϕ is being calculated. Thus $t-r/c$ is the moment at which a signal would require to be sent from the element dv in order to reach P at the given time t. For this reason (20) and (21) are usually called **retarded potentials** and $t-r/c$ is the **retarded time.**

§ 114. Potential of a moving electron

If we may suppose that each element of volume in (20) and (21) contributes to ϕ and **A** independently of all other

* For a full discussion of this, see Coulson, *Waves*, Oliver and Boyd Ltd., § 88.

elements, we may calculate ϕ and **A** for an electron of charge e and velocity $\mathbf{v}$ by putting $\rho\,dv = e$ and $\mathbf{j}\,dv = e\mathbf{v}$, and noting that in the denominator r becomes the retarded distance $[r]$. Thus

$$\phi = \frac{e}{[r]}, \ \mathbf{A} = \frac{e[\mathbf{v}]}{[r]} \quad . \quad . \quad . \quad (22)$$

From these, by differentiation, it is possible to calculate the associated electric and magnetic fields and the rate of radiation of energy.

§ 115. Recent developments

It is beyond the scope of this book to discuss the further developments of the fundamental principles of electromagnetism. Such studies mostly fall into one of two categories. First, there are attempts to co-ordinate the electric and magnetic fields in a comprehensive scheme in which a four-dimensional analysis is needed.* This soon involves the theory of relativity, but the complete formulation of a unified field theory in which relativity, gravitation and electromagnetism are seen to be properly related, has not yet been achieved. Then, secondly, in order to understand many of the experimental results which lie beyond the power of our previous analysis, it is necessary to use the methods of Quantum Theory to study in more detail the nature and structure of a single atom, and the forces between atoms. This theory is able to give a convincing account of the meaning and numerical values of the hitherto empirical constants σ, μ and K; it also explains the colour, or characteristic spectrum, of an atom or molecule, and many otherwise puzzling phenomena. But with this brief introduction to the elaborate field of modern Theoretical Physics we must leave the reader.

* For an introduction to this, see Rutherford, *Vector Methods*, Chapter VIII.

§ 116. Examples

1. According to (4) the total current **J** involves a polarisation current. Show by analysis similar to that in § 25 that the magnitude of this must be $\partial\mathbf{P}/\partial t$.

2. Obtain the differential equation (11) for the electric vector **E** in the case where the volume charge ρ is zero.

3. Verify the result stated in (10).

4. In a certain electrical problem $E_x = 0$, $E_y = 0$, and $E_z = a \cos nx \cos nct$. It is given that $\mathbf{H} = 0$ when $t = 0$ and $K = \mu = 1$, $\rho = \sigma = 0$. Prove that $H_x = 0$, $H_z = 0$, and that $H_y = -a \sin nx \sin nct$. Verify that there is no mean flux of energy in this problem, which is a case of stationary waves.

5. A large sphere of radius b is made of material of conductivity σ and dielectric constant K. At time $t = 0$ a charge Q_0 is uniformly distributed over the surface of a small concentric sphere of radius a. Show that at time t the charge Q on the inner sphere is given by $Q = Q_0 e^{-nt}$, where $n = 4\pi\sigma/K$. Show further that the total Joule heat loss during the discharge is $\dfrac{Q_0^2}{2K}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$. Verify that this is also the decrease in electrostatic energy, electrostatic units being used throughout.

6. Show that for any finite system the Poynting Vector tends to zero at infinity so fast that its integral over the sphere at infinity is zero. This shows that the total electromagnetic energy remains constant.

7. The formula for the flow of energy $\dfrac{c}{4\pi}(\mathbf{E}\times\mathbf{H})$ was proved in § 111 for the case of a closed surface only. Verify that it is not true for an open surface in the case where **E** and **H** are two constant perpendicular fields; but it is still true even in this case for any closed surface.

8. Starting with the formula (14) for the electromagnetic energy U in a given volume, show from Maxwell's equations that

$$\frac{\partial U}{\partial t} = -\frac{c}{4\pi}\int (\mathbf{E}\times\mathbf{H}).d\mathbf{S} - c\int \mathbf{j}.\mathbf{E}\, dv.$$

Interpret the two terms on the right-hand side and notice that this provides an alternative derivation of the Poynting Vector.

9. Obtain equations (16) and (17) for the potentials **A** and ϕ.

10. Show that the potentials **A** and ϕ (§ 107) which describe the plane wave (10) are

$$\phi = -ax \cos p\left(t - \frac{z}{V}\right),\ A_x = A_y = 0,\ A_z = -\frac{acx}{V}\cos p\left(t - \frac{z}{V}\right)$$

11. It is sometimes convenient to replace the conditions (viii)-(x) in § 107 by alternative conditions.

$$\mathbf{B} = \text{curl}\,\mathcal{A},\ \mathbf{E} = -\frac{1}{c}\frac{\partial\mathcal{A}}{\partial t} - \text{grad}\,\psi,\ \text{div}\,\mathcal{A} = -4\pi\sigma\mu\psi - \frac{K\mu}{c}\frac{\partial\psi}{\partial t}.$$

Show that if we may put $\rho = 0$ the new potentials $\mathcal{A}$ and ψ satisfy the equation

$$\nabla^2 = \frac{4\pi\sigma\mu}{c}\frac{\partial}{\partial t} + \frac{K\mu}{c^2}\frac{\partial^2}{\partial t^2}.$$

These equations are important because they are the same as the differential equation (11) for **E** and **H**.

12. Show that if in the previous question we put $\psi = -\text{div}\,\mathbf{Z}$, then $\mathcal{A} = 4\pi\sigma\mu\,\mathbf{Z} + \dfrac{K\mu}{c}\dfrac{\partial\mathbf{Z}}{\partial t}$. Show also that we may now put $\mathbf{E} = \text{curl curl}\,\mathbf{Z}$ where **Z** satisfies the same equation as $\mathcal{A}$ and ψ, viz.,

$$\nabla^2\mathbf{Z} = \frac{4\pi\sigma\mu}{c}\frac{\partial\mathbf{Z}}{\partial t} + \frac{K\mu}{c^2}\frac{\partial^2\mathbf{Z}}{\partial t^2}.$$

Z is called the **Hertzian Vector**. We may notice that all properties of the electromagnetic wave are contained in the one differential equation for **Z**.

13. Show that it is possible to solve equation (11) for waves of frequency $p/2\pi$ in a conducting medium by putting $E_y = E_z = 0$, $E_x = a\, e^{ip(t-qz)}$, where q is given in terms of p by the equation

$$q^2 = \frac{K\mu}{c^2}\left\{1 - \frac{4\pi\sigma c}{Kp}\, i\right\}.$$

By taking the real part of E_x show that a harmonic wave cannot be propagated in such a medium without absorption.

14. In the previous question $4\pi\sigma c$ is very much greater than Kp. Show that $q = \gamma(1-i)$, where $\gamma^2 = 2\pi\sigma\mu/pc$. Deduce

that H_y is the only non-vanishing component of $\mathbf{H}$ and that H_y and E_z have a $\pi/4$ phase difference.

15. Verify that a particular solution of equation (12) is $j_x = j_y = 0$, $j_z = a\,e^{-p\gamma x}\cos p(t-\gamma x)$, where $\gamma^2 = 2\pi\sigma\mu/pc$. Interpret this in terms of the skin effect in a semi-infinite conductor bounded by the plane $x = 0$.

16. Show that the magnetic field energy $\frac{1}{8\pi}\int \mathbf{B}.\mathbf{H}\,dv$ confined in a given volume may be written in the form

$$\tfrac{1}{2}\int \mathbf{A}.\mathbf{j}\,dv + \frac{1}{8\pi c}\int \mathbf{A}.\frac{\partial \mathbf{D}}{\partial t}\,dv - \frac{1}{8\pi}\,\text{flux}\,(\mathbf{H}\times\mathbf{A}).$$

Deduce that in a finite system of quasi-steady currents the total magnetic field energy is simply $\frac{1}{2}\int \mathbf{A}.\mathbf{j}\,dv$, which, if μ is constant, may be written as $\frac{1}{2}\mu\int \frac{(\mathbf{j}.\mathbf{j}')}{r}\,dv\,dv'$. In this integral $\mathbf{j}$ and $\mathbf{j}'$ are the values of $\mathbf{j}$ at dv and dv', r is the distance between dv and dv' and the integration covers the whole of space for both dv and dv'.

17. Taking in the previous question the special case of a pair of closed linear circuits carrying currents i_1 and i_2, show that the extra field energy due to their interaction is $\mu i_1 i_2 \int \frac{\mathbf{ds}_1.\mathbf{ds}_2}{r}$, i.e. $M_{12}i_1i_2$. (cf. § 64). The total magnetic field energy has thus been shown to be $\frac{1}{2}L_1i_1^2 + M_{12}i_1i_2 + \frac{1}{2}L_2i_2^2$.

18. Current i flows in a straight wire whose section is a circle of radius a. Show that the magnetic field energy inside the wire is $\mu i^2/4$ per unit length. Deduce that the contribution to the self-induction that arises from the energy stored inside the wire is $\frac{1}{2}\mu$ per unit length. (Use the field given in § 55, question (4).) This is called the **internal self-inductance.** The external self-inductance may be obtained by regarding the current as all concentrated along the central line. This provides a method of calculating self-inductances.

19. The total charge Q on a conducting sphere of radius a is made to vary so that $Q = 4\pi a^2\sigma$, where $\sigma = 0$ for $t<0$ and $\sigma = \sigma_0 \sin pt$ for $t>0$. Show that if $K = \mu = 1$, then the potential ϕ at a distance R $(R>a)$ from the centre of the sphere, is given by

$$ct<R-a,\ \phi = 0,$$

$$R-a<ct<R+a,\ \phi = \frac{2\pi ac\sigma_0}{pR}\left\{1-\cos p\left(t-\frac{R-a}{c}\right)\right\}$$

$$R+a<ct,\ \phi = \frac{4\pi ac\sigma_0}{pR}\sin\frac{pa}{c}\sin p\left(t-\frac{R}{c}\right).$$

20. Show that if $u = t-r/c$ and $g(u)$ is any function of u, then a possible solution of equation (17) in which $\rho = 0$, $K = \mu = 1$, and ϕ depends only on the time t and the distance r from the origin is, $\phi = g(u)/r$.

21. Verify that a possible solution of equations (16) and (17) for the potentials in free space is obtained by choosing any function $f(u)$ of the variable $u = t-r/c$, and writing

$$A_x = A_y = 0,\ A_z = \frac{f'(u)}{cr},\ \phi = \left\{\frac{f'(u)}{cr} + \frac{f(u)}{r^2}\right\}\frac{z}{r}.$$

This is Hertz's solution of the field due to an electric dipole (or **Hertzian oscillator**), whose moment is $f(t)$ pointing along the z axis. $f(t)$ is usually an oscillating periodic function of t, and by this means it is possible to discuss radiation from short aerials. The solution can also be used to calculate the radiation from an accelerated electron.

22. Show that in the previous question the rate of flow of energy across a large sphere is $\frac{2}{3c^3}[f''(u)]^2$.

23. An electron of charge e is moving with constant velocity $\mathbf{v}$ along a straight line. P is a point such that the line joining P to the electron is of length r and makes an angle θ with the direction of motion. Show that if v is much less than c, we may write

$$\phi_P = \frac{e}{r}\left(1+\frac{v}{c}\cos\theta\right),\ \mathbf{A}_P = \frac{e\mathbf{v}}{r}\left(1+\frac{v}{c}\cos\theta\right).$$

24. Use the formulæ of the last question to calculate the electric and magnetic fields at the point P.

CHAPTER XIV

UNITS AND DIMENSIONS

§ 117. Dimensions

WE have left till now any discussion of units and dimensions. This is intentional, because the basic principles previously outlined are completely independent of the particular practical units concerned, and it is better therefore not to confuse matters of principle with questions of practice.

There is, however, one important matter to which (§ 45) reference has already been made. We have from time to time used two completely independent systems of measurement, the e.m.u. and e.s.u. These are each self-consistent, but they are not the same and (as we have developed them) actually predict different dimensions for the same quantity.

Thus, considering first the e.s.u., we remember that the force between two charges Q_1 and Q_2 *in vacuo* is $F = Q_1Q_2/r^2$. Since the dimensions of force are MLT^{-2}, this gives $Q = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$. But current is rate of flow of charge and must therefore be $M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$; current density is current per unit area of section and must be $M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-2}$. In this way the dimensions of all the required quantities may be obtained, on the supposition that the dielectric constant is a pure number. But it may be objected that the real law of force is $F = Q_1Q_2/Kr^2$, and that by putting $K = 1$ in free space we do not remove its dimensional character. In such a case we must develop all our dimensional analysis in terms of MLT and K. The results of both calculations are shown in table 1 at the end of the chapter. The student is strongly urged to draw up this table for himself, filling in the last five columns without consulting the book.

Similarly we can deal with the e.m.u. Here we can start either with a law of force between poles p_1 and p_2 given by $F = p_1p_2/\mu r^2$, or, if we do not wish to introduce such unobservable entities as isolated magnetic poles, we may use the mutual potential energy $M_{12}i_1i_2$ (§ 53) between two currents. The resulting dimensions, with and without the inclusion of the permeability μ, are also shown in Table 1.

§ 118. Ratio of e.m.u. and e.s.u.

Reference to Table 1 shows that if we neglect any possible dimensions in μ and K, the e.s.u. and e.m.u. always differ by a power of LT^{-1}, i.e. a velocity. Thus the $\dfrac{\text{e.s.u. of charge}}{\text{e.m.u. of charge}}$ has dimensions LT^{-1} and the $\dfrac{\text{e.s.u. of resistance}}{\text{e.m.u. of resistance}}$ has dimensions $(LT^{-1})^{-2}$. This ratio which is found numerically (see § 109) to have the same value as the velocity of light in free space, is represented by **c**, with a numerical value approximately 3×10^{10} cms. per sec. We show in the last column of Table 1 the ratio of the e.s.u. and the e.m.u. for the various quantities. Thus, from the Table, we see that 1 e.s.u. of electric field **E** equals 3×10^{10} e.m.u. of electric field. In this way Table 1 may be used to convert from one system to the other.

On the other hand, if we take into account the dimensions of K and μ we are obliged to use four and not three fundamental units. We may choose M and any three of L, T, K, μ. This follows from Table 1, in which it is easily verified from the last five columns that when K and μ are included the e.m.u. and e.s.u. predict the same dimensions for all quantities if $[K][\mu] = [\text{velocity}]^{-2}$.

One possible objection to the use of all such systems as these is that fractional exponents occur in the dimensions of most quantities. It is possible, though we shall not discuss this further, to avoid all fractional indices by working in terms of M, L, T and Q as the four basic quantities.

Q

TABLE 1—*Dimensions in e.s.u. and e.m.u.*

Quantity.	Symbol.	e.s.u. K-dimensionless.	e.m.u. μ-dimensionless.	e.s.u. including K	e.m.u. including μ	$\frac{\text{One e.s.u.}}{\text{One e.m.u.}}$
Capacity	C	L	$L^{-1}T^2$	K	μ^{-1}	c^{-2}
Charge	Q	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}$	$K^{\frac{1}{2}}$	$\mu^{-\frac{1}{2}}$	c^{-1}
Conductivity	σ	T^{-1}	$L^{-2}T$	K	μ^{-1}	c^{-2}
Current	i	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	$K^{\frac{1}{2}}$	$\mu^{-\frac{1}{2}}$	c^{-1}
Current vector	$\mathbf{j}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-2}$	$M^{\frac{1}{2}}L^{-\frac{3}{2}}T^{-1}$	$K^{\frac{1}{2}}$	$\mu^{-\frac{1}{2}}$	c^{-1}
Dielectric constant	K	...	$L^{-2}T^2$	K	μ^{-1}	c^{-2}
Displacement	$\mathbf{D}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$	$M^{\frac{1}{2}}L^{-\frac{3}{2}}$	$K^{\frac{1}{2}}$	$\mu^{-\frac{1}{2}}$	c^{-1}
E.m.f.	$\mathcal{E}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$	$K^{-\frac{1}{2}}$	$\mu^{\frac{1}{2}}$	c
Electric field	$\mathbf{E}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}$	$K^{-\frac{1}{2}}$	$\mu^{\frac{1}{2}}$	c
Magnetic field	$\mathbf{H}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$	$K^{\frac{1}{2}}$	$\mu^{-\frac{1}{2}}$	c^{-1}
Magnetic flux	N	$M^{\frac{1}{2}}L^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$	$K^{-\frac{1}{2}}$	$\mu^{\frac{1}{2}}$	c
Inductance	L, M	$L^{-1}T^2$	L	K^{-1}	μ	c^2
Magnetic induction	$\mathbf{B}$	$M^{\frac{1}{2}}L^{-\frac{3}{2}}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$	$K^{-\frac{1}{2}}$	$\mu^{\frac{1}{2}}$	c
Magnetic moment	$\mathbf{m}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}$	$M^{\frac{1}{2}}L^{\frac{5}{2}}T^{-1}$	$K^{-\frac{1}{2}}$	$\mu^{\frac{1}{2}}$	c
Magnetisation	$\mathbf{I}, \mathbf{M}$	$M^{\frac{1}{2}}L^{-\frac{3}{2}}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$	$K^{-\frac{1}{2}}$	$\mu^{\frac{1}{2}}$	c
Magnetostatic potential	Ω	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	$K^{\frac{1}{2}}$	$\mu^{-\frac{1}{2}}$	c^{-1}
Permeability	μ	$L^{-2}T^2$	...	K^{-1}	μ	c^2
Polarisation	$\mathbf{P}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$	$M^{\frac{1}{2}}L^{-\frac{3}{2}}$	$K^{\frac{1}{2}}$	$\mu^{-\frac{1}{2}}$	c^{-1}
Pole strength	p	$M^{\frac{1}{2}}L^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$	$K^{-\frac{1}{2}}$	$\mu^{\frac{1}{2}}$	c
Electrostatic potential	V, ϕ	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$	$K^{-\frac{1}{2}}$	$\mu^{\frac{1}{2}}$	c
Magnetic vector potential	$\mathbf{A}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	$K^{-\frac{1}{2}}$	$\mu^{\frac{1}{2}}$	c
Resistance	R	$L^{-1}T$	LT^{-1}	K^{-1}	μ	c^2
Specific resistance	τ	T	L^2T^{-1}	K^{-1}	μ	c^2

§ 119. Gaussian units

It is possible to work entirely in e.s.u. or entirely in e.m.u. But in that case the powers of c in Maxwell's equations differ from those given in § **107**. It is more convenient to use mixed units, sometimes called Gaussian units, instead. Here magnetic quantities are measured in e.m.u. and electrostatic quantities in e.s.u. Maxwell's equations then show a symmetry in powers of c that makes them easier to remember. In this system some writers take the current $\mathbf{j}$ in e.s.u. and others (as in this book) in e.m.u. Our division is:

e.s.u.—charge, electric field, displacement, capacity and related quantities;

e.m.u.—current, magnetic field, induction, resistance and related quantities.

Note that with these units (6) on p. 75 becomes $\operatorname{div} \mathbf{j} = -\frac{1}{c}\frac{\partial \rho}{\partial t}$.

Also, if σ is in e.s.u., $\mathbf{j} = \sigma\mathbf{E}/c$; if σ is in e.m.u., $\mathbf{j} = c\sigma\mathbf{E}$.

§ 120. Rationalised units

In order to avoid the factor 4π which is continually appearing in our formulæ various rationalised systems have been introduced by means of which it is removed. This is achieved by inserting a factor 4π in the definitions of dielectric constant and permeability. As a result, in free space K and μ are no longer unity; we generally write them K_0 and μ_0. If we are using rationalised Gaussian units, K_0 and μ_0 have the value $1/4\pi$ and 4π respectively. The most important rationalised system is the **Giorgi** system. This leaves unchanged the practical units of ohm, Coulomb and volt (see § **122** and Table 4 for further details.)

Note to Table 1.

Column headed "e.s.u. K-dimensionless" gives the dimensions when the law of force is $F = Q_1Q_2/r^2$.

Column headed "e.m.u. μ-dimensionless" gives the dimensions when the law of force between poles is $F = p_1p_2/r^2$, i.e. when the mutual potential energy of two small magnets is $(\mathbf{m}_1 . \mathbf{m}_2)/r^3 - 3(\mathbf{m}_1 . \mathbf{r})(\mathbf{m}_2 . \mathbf{r})/r^5$.

Column headed "e.s.u. including K" gives the degree of K by which the "K-dimensionless" column must be multiplied when the law of force is Q_1Q_2/Kr^2.

Column headed "e.m.u. including μ" gives the degree of μ by which the "μ-dimensionless" column must be multiplied when the law of force is $p_1p_2/\mu r^2$.

§ 121. Practical units—c.g.s. system

The absolute e.s.u. and e.m.u. are not convenient in magnitude for ordinary measurements, and we therefore

TABLE 2—*Practical c.g.s. units*

Quantity.	Symbol.	Name of Practical c.g.s. unit.	Measure in e.s.u.	Measure in e.m.u.
Capacity	C	Farad	$10^{-9}\,c^2$	10^{-9}
Charge	Q	Coulomb	$10^{-1}\,c$	10^{-1}
Conductivity	σ	Mho per cm.	$10^{-9}\,c^2$	10^{-9}
Current	i	Ampere	$10^{-1}\,c$	10^{-1}
Current vector	$\mathbf{j}$	Amp. per sq. cm.	$10^{-1}\,c$	10^{-1}
Dielectric constant	K	4π - farad per cm.	$10^{-9}\,c^2$	10^{-9}
Displacement	$\mathbf{D}$	4π - coulombs per sq. cm.	$10^{-1}\,c$	10^{-1}
E.m.f.	$\mathcal{E}$	Volt	$10^{8}\,c^{-1}$	10^{8}
Electric field	$\mathbf{E}$	Volt per cm.	$10^{8}\,c^{-1}$	10^{8}
Magnetic field	$\mathbf{H}$	Oersted	c	1
Magnetic flux	N	Maxwell	c^{-1}	1
Inductance	L, M	Henry	$10^{9}\,c^{-2}$	10^{9}
Magnetic induction	$\mathbf{B}$	Gauss	c^{-1}	1
Magnetic moment	$\mathbf{m}$	Maxwell-cm.	$\frac{1}{4\pi}c^{-1}$	$\frac{1}{4\pi}$
Magnetisation	$\mathbf{I}, \mathbf{M}$	Maxwell per sq. cm.	$\frac{1}{4\pi}c^{-1}$	$\frac{1}{4\pi}$
Magnetostatic potential	Ω	Gilbert	c	1
Permeability	μ	Gauss per oersted	c^{-2}	1
Polarisation	$\mathbf{P}$	Coulomb per sq. cm.	$10^{-1}\,c$	10^{-1}
Pole strength	p	Maxwell	$\frac{1}{4\pi}c^{-1}$	$\frac{1}{4\pi}$
Electrostatic potential	V, ϕ	Volt	$10^{8}\,c^{-1}$	10^{8}
Magnetic vector potential	$\mathbf{A}$	Gauss-cm.	c^{-1}	1
Resistance	R	Ohm	$10^{9}\,c^{-2}$	10^{9}
Specific resistance	τ	Ohm-cm.	$10^{9}\,c^{-2}$	10^{9}

introduce practical units, the coulomb, volt, etc., based on the e.m.u. Table 2 shows the relation between the practical and

absolute c.g.s. units. These new units are consistent among themselves. Thus:

1 amp. = 1 coulomb per second,
1 farad = 1 coulomb per volt,
1 ohm = 1 volt per amp., etc.

A relation often worth remembering is that 1 e.s.u. of potential is the same as 300 volts.

Since these units are derived from an absolute definition of charge, current, etc., we may refer to them as the absolute amp., the absolute ohm, etc. Until recently absolute measurements of this kind have proved extremely difficult to make; accordingly, for practical convenience and by international agreement, the ohm, amp. and volt were defined in terms of simpler definite physical apparatus. These were called the international units (int. units). Thus the int. amp. was that current which deposited 0·001118 gram of silver per second in a voltameter; and the int. ohm was the resistance of a certain definite column of mercury. These units were introduced because, for example, it is much simpler to measure current by a voltameter than by using a Kelvin current balance to determine the force between two coils. In fact, the methods by which the int. units were defined were devised at an early stage before the theoretical relations between the various quantities were properly understood. As a result the absolute and international units were not quite the same. Thus:

1 int. ohm was about 1·0005 abs. ohm.

This difference, and the corresponding differences in the other units, were small, and troublesome. Although they were insignificant for most ordinary purposes, they were an annoying necessity in precision work. In addition to this, with the passage of time, devices for the measurement of absolute magnitudes were improved. The result was that, on 1st January 1948, by international agreement, the older units (of 1908) were abandoned: now only the absolute

TABLE 3—*Relation between m.k.s. and c.g.s. Mechanical Units*

Quantity.	M.k.s. System.		C.g.s. System.
Length	1 meter	=	10^2 cms.
Mass	1 kilogram	=	10^3 gms.
Time	1 second	=	1 second
Energy	1 joule	=	10^7 ergs
Force	1 newton	=	10^5 dynes
Power	1 watt	=	10^7 ergs per sec.

TABLE 4—*Practical Rationalised m.k.s. or Giorgi Units*

Quantity.	Symbol.	Name.	Measure in Unrationalised e.s.u.	Measure in Unrationalised e.m.u.
Capacity . . .	C	Farad	$10^{-9} c^2$	10^{-9}
Charge . . .	Q	Coulomb	$10^{-1} c$	10^{-1}
Conductivity . .	σ	Mho per meter	$10^{-11} c^2$	10^{-11}
Current . . .	i	Ampere	$10^{-1} c$	10^{-1}
Current vector . .	j	Ampere per sq m.	$10^{-5} c$	10^{-5}
Dielectric constant .	K	Farad per meter	$4\pi\ 10^{-11} c^2$	$4\pi\ 10^{-11}$
Displacement . .	D	Coulomb per sq. m.	$4\pi\ 10^{-5} c$	$4\pi\ 10^{-5}$
E.m.f. . . .	$\mathcal{E}$	Volt	$10^8 c^{-1}$	10^8
Electric field . .	E	Volt per meter	$10^6 c^{-1}$	10^6
Magnetic field . .	H	Ampere-turn per m.	$4\pi\ 10^{-3} c$	$4\pi\ 10^{-3}$
Magnetic flux . .	N	Weber	$10^8 c^{-1}$	10^8
Inductance . .	L, M	Henry	$10^9 c^{-2}$	10^9
Magnetic induction .	B	Weber per sq. m.	$10^4 c^{-1}$	10^4
Magnetic moment .	m	Weber-meter	$\frac{1}{4\pi} 10^{10} c^{-1}$	$\frac{1}{4\pi} 10^{10}$
Magnetisation . .	I, M	Weber per sq. m.	$\frac{1}{4\pi} 10^4 c^{-1}$	$\frac{1}{4\pi} 10^4$
Magnetostatic potential	Ω	Ampere-turn	$4\pi\ 10^{-1} c$	$4\pi\ 10^{-1}$
Permeability . .	μ	Henry per meter	$\frac{1}{4\pi} 10^7 c^{-2}$	$\frac{1}{4\pi} 10^7$
Polarisation . .	P	Coulomb per sq. m.	$10^{-5} c$	10^{-5}
Pole strength . .	p	Weber	$\frac{1}{4\pi} 10^8 c^{-1}$	$\frac{1}{4\pi} 10^8$
Electrostatic potential	V, ϕ	Volt	$10^8 c^{-1}$	10^8
Magnetic vector potential	A	Weber per meter	$10^6 c^{-1}$	10^6
Resistance . .	R	Ohm	$10^9 c^{-2}$	10^9
Specific resistance .	τ	Ohm-meter	$10^{11} c^{-2}$	10^{11}

units remain. But it is important to know of the existence of the former international units because they appear so frequently in older textbooks.

§ 122. m.k.s. units

We have hitherto used the centimetre, gram and second (c.g.s. system) as our fundamental mechanical units. But a more convenient system for many purposes is obtained if we take as fundamental units the meter, kilogram and second (m.k.s. system); and in fact the c.g.s. system has been largely superseded by the m.k.s. system in recent years. The relation between the most common mechanical quantities in the c.g.s. and m.k.s. systems is shown in Table 3, which enables us to convert easily from one to the other.

The practical electrical units are easily expressed in terms of the m.k.s. system by combining Tables 2 and 3. However, it is worth while showing this relationship quite precisely; thus Table 4 gives the practical unit in the m.k.s. system and its relation to the ordinary unrationalised e.s.u. and e.m.u.; the rationalised system used in the electrical m.k.s. units is the Giorgi system (see § 120) which introduces a factor 4π in K and μ, but leaves the ohm and the volt unchanged. In July 1950, by international agreement, it was decided to recommend the adoption of the total rationalised Giorgi system. This recommendation, however, is still subject to ratification though its use is now becoming almost standard practice in original papers published in electrical engineering journals.

Finally, in Table 5, we give the numerical values of some related and useful physical quantities.

TABLE 5—*Some Useful Constants*

Velocity of light in free space $= c$	$= 2 \cdot 998 \times 10^{10}$ cms. per sec.
Charge on an electron $= e$	$= 4 \cdot 803 \times 10^{-10}$ e.s.u.
Mass of an electron $= m$	$= 9 \cdot 110 \times 10^{-28}$ gms.
e/m for an electron (the specific charge)	$= 1 \cdot 76 \times 10^7$ abs. e.m.u. per gm.
Mass of a hydrogen atom $= M$	$= 1 \cdot 660 \times 10^{-24}$ gms.
Horizontal component of the earth's magnetic field	$= 0 \cdot 18$ oersted (approx.)
Field between poles of a large electro-magnet	$= 10{,}000$ oersteds

Conductivity of copper (a metallic conductor)	$=5{\cdot}8\times10^5$ mho per cm.
Conductivity of shellac (insulator)	$=10^{-16}$ mho per cm.
Permeability of soft iron	$=50$ to 1000 gauss per oersted
Permeability of gold	$=$between 1 and 3 times 10^{-6} gauss per oersted
Dielectric constant of glass	$=5$ to 10 4π-farad per cm.
Planck's constant $=h$	$=6{\cdot}62\times10^{-27}$ erg-sec.
Avogadro's number = number of molecules in a gram-molecule	$=6{\cdot}024\times10^{23}$
Permeability of free space (§ 120) $=\mu_0$	$=4\pi\times10^{-7}$ henry per metre
Dielectric constant of free space (§ 120) $=K_0$	$=8{\cdot}854\times10^{-12}$ farad per metre

NOTE ON THE SIMILARITIES BETWEEN E, D, B AND H

THE electromagnetic field is defined by the four basic vectors **E**, **D**, **B** and **H**. Between these vectors there are obvious similarities, whose existence enables us, for example, to "carry over" much of the formalism of electrostatics to problems in magnetism and the flow of current. Our whole discussion of potential problems in Chapters IX and X was based on this parallelism.

The treatment which we have used in this book is the traditional one in which the magnetic vectors **H** and **B** are supposed to correspond with the electric vectors **E** and **D** respectively. This correspondence is expressed by the relations

$$\mathbf{D}=K\mathbf{E} \quad , \quad \mathbf{B}=\mu\mathbf{H}. \quad . \quad . \quad . \quad (1)$$

The question which we now want to discuss is whether **H** does really correspond to **E**, and **B** to **D**, or whether (1) should not rather be written

$$\mathbf{D}=K\mathbf{E} \quad , \quad \mathbf{H}=\frac{1}{\mu}\mathbf{B}. \quad . \quad . \quad . \quad (2)$$

In this form the parallelism is between **H** and **D**, and between **B** and **E**. It will soon be recognised that in the form (2) a good deal of the mathematical formalism of electrostatics still carries over to the other branches of our subject, so that the analysis of Chapters IX and X still preserves its general

applicability, and Maxwell's equations of the electromagnetic field are supposed, of course, to remain unchanged. It is true that there would be a few changes of sign—mathematically trivial—in the "carry over," as for example when we compare (Chapters IV and VIII)

$$\mathbf{D}=\mathbf{E}+4\pi\mathbf{P} \quad , \quad \mathbf{H}=\mathbf{B}-4\pi\mathbf{M},$$

instead of

$$\mathbf{D}=\mathbf{E}+4\pi\mathbf{P} \quad , \quad \mathbf{B}=\mathbf{H}+4\pi\mathbf{M}.$$

The matter is important because it is very desirable that we should be clear regarding which are the fundamental electric and magnetic vectors, and because, in more advanced discussions of the energy than are possible in this book, the distinction becomes extremely significant. As we shall see, the parallelism represented by (2) is, in fact, preferable to that represented by (1). It is the purpose of this note to point out the justification for this more modern interpretation, even though none of our major results and formulæ are thereby affected.

In free space, where $\mathbf{D}\equiv\mathbf{E}$, and $\mathbf{B}\equiv\mathbf{H}$, there is no distinguishing between the two descriptions (unless we are using some of the other systems of units referred to in § 120, in which K_0 and μ_0 are not equal to unity). But in dielectric media we recognise **E** as fundamental for the determination of the force exerted on a charge. We should therefore expect that the magnetic vector corresponding to **E** in electrostatics would be that vector which measures force in the presence of magnetic media. Now from (32) in § 62 we see that the total force on a moving charge is

$$e\mathbf{E}+\frac{e}{c}\mathbf{v}\times\mathbf{B},$$

and from (31) in the same section, the force on a circuit carrying a current i is

$$\frac{i}{c}\int d\mathbf{s}\times\mathbf{B}.$$

Both of these suggest that **B** is the fundamental force vector in magnetism, and should be taken to correspond to **E**. Several other arguments reinforce this view. Thus :—

(i) In Maxwell's equations we find that curl $\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$, showing that **E** and **B** are related together. So also are **H** and **D** in one of the other equations ;

(ii) In fundamental studies of the electromagnetic field, it turns out that the components of **B** and of $i\mathbf{E}$ form the six elements of an antisymmetrical matrix of order 4,* enabling a particularly neat and suggestive expression of the set of field equations to be given ;

(iii) In considering the relativistic transformations that take place when one set of axes is moving relative to another, it appears that **B** as measured by an observer in one frame of axes is related to a combination of **B** and **E** as measured by an observer in the other frame ;

(iv) In the analogies which exist between electrodynamics and ordinary mechanics, it appears that a Lagrangian function can be introduced in electrodynamics, and that this function can be used in a similar way to its counterpart in dynamical problems.† The form of this Lagrangian is $\frac{1}{8\pi}\int(\mathbf{B}^2-\mathbf{E}^2)dv$.

All these arguments suggest that the more fundamental point of view is expressed by (2) rather than by (1). Present custom as revealed in the larger treatises on electricity and magnetism tends to follow this line of thought, introducing the magnetic field (analogously to the electric field) in permeable media in terms of the force vector **B**. Older tradition, as we have said, was in favour of (1).

* A brief account is in Rutherford, *Vector Methods*, Oliver and Boyd Ltd., Chapter VIII.

† See e.g. Rutherford, *Mechanics*, Oliver and Boyd Ltd.

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